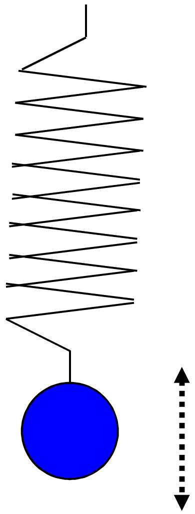
Oscillations

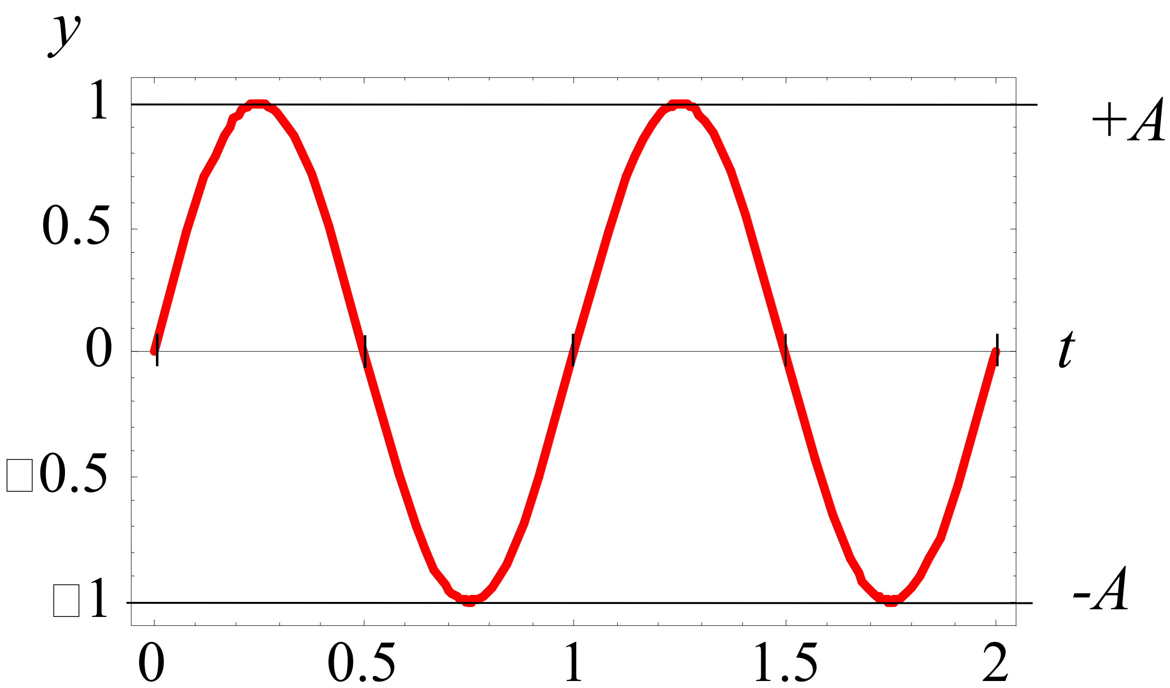
Free non-damped oscillations.
Superposition of oscillations, beats.
Free damped oscillations. Resonance.

Harmonic motion

Periodic and space limited motion

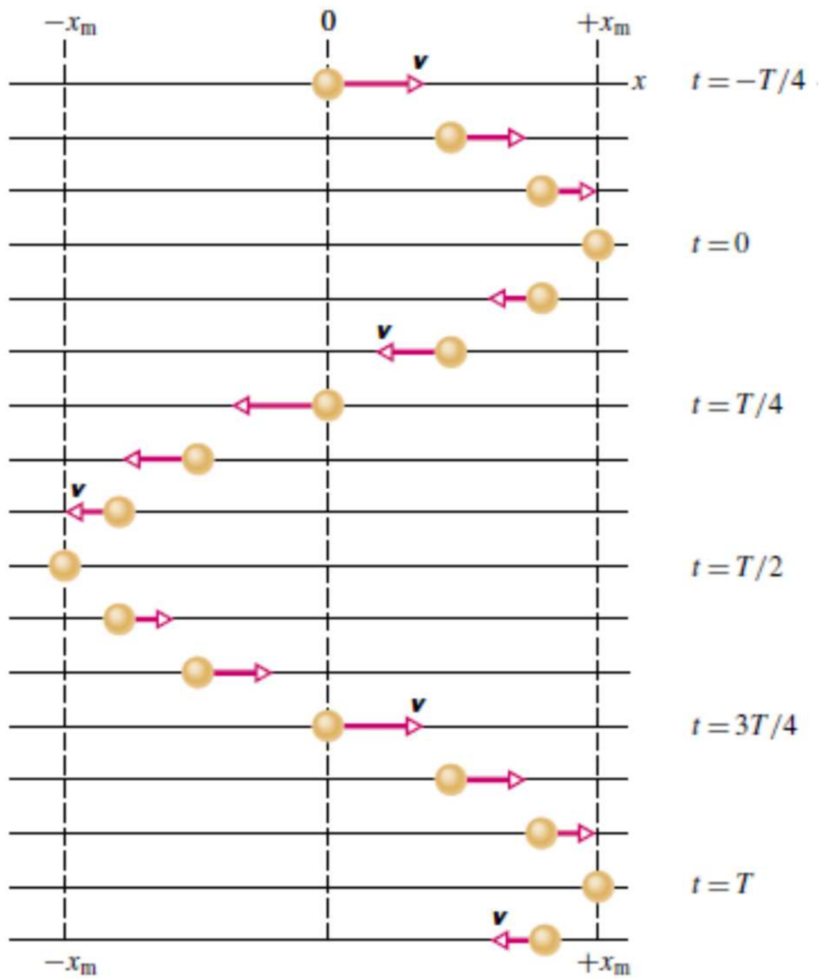


$$y = A\sin(\omega t + \varphi)$$



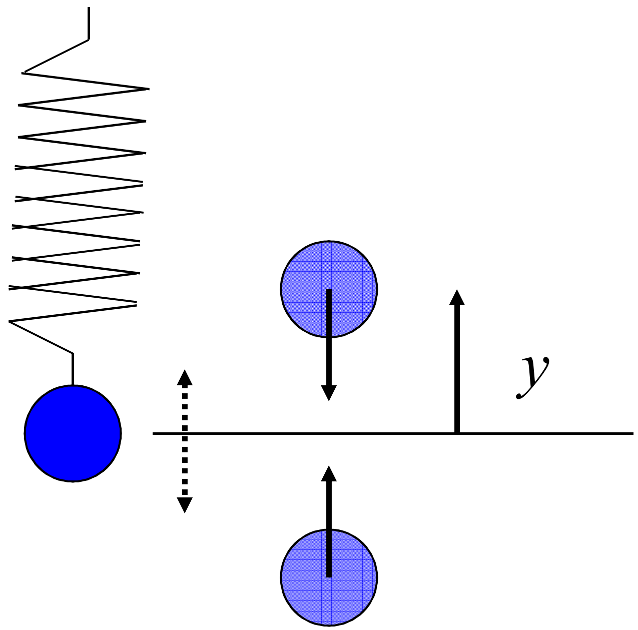
Harmonic motion

Displacement and velocity change magnitude and direction



Restoring force

(Linear) restoring force



$$F = -ky$$

k … spring constant

y … displacement

Type of oscillations

- Periodic
- Non-periodic

- Free
- Forced

- Damped
- Non-damped

- Linear
- Non-linear

etc.

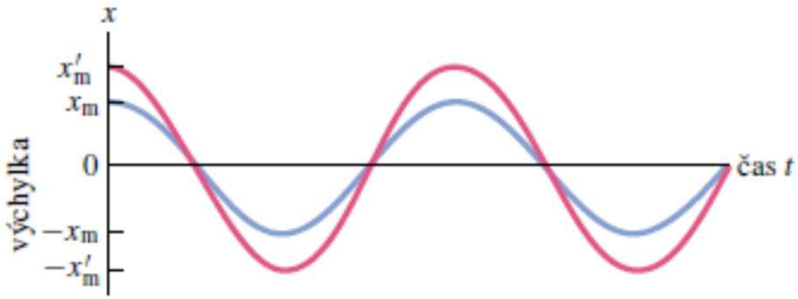
Non-damped oscillation motion parameters

Equation of motion solution $y = A\sin(\omega t + \varphi)$

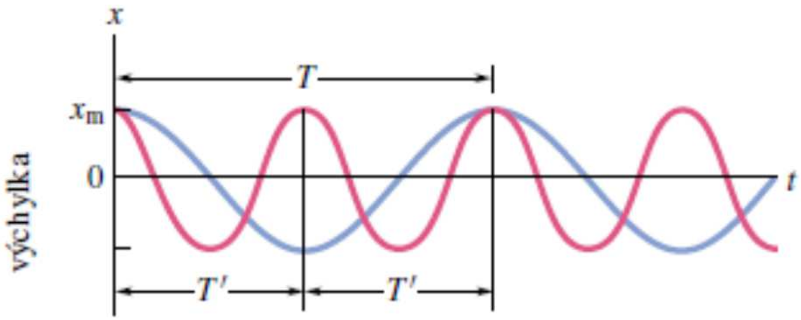
- Displacement - y
- Amplitude - A
- Angular frequency – ω, frequency - f
- Period – $\omega = \dfrac{2\pi}{T} = 2\pi f$
- Initial phase - φ

Oscillation parameters

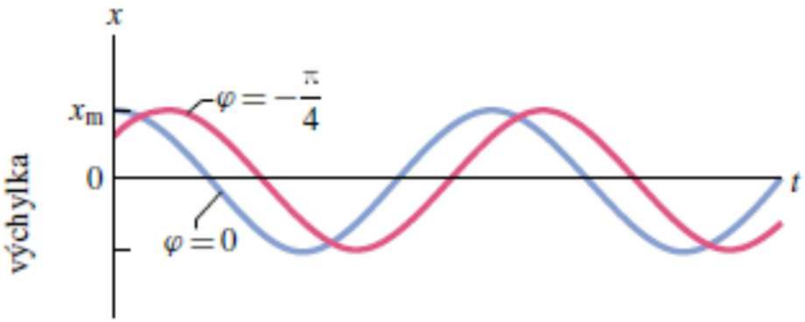
Amplitude



Period



Initial phase



Equation of motion for harmonic oscillations

Equation of motion for restoring force

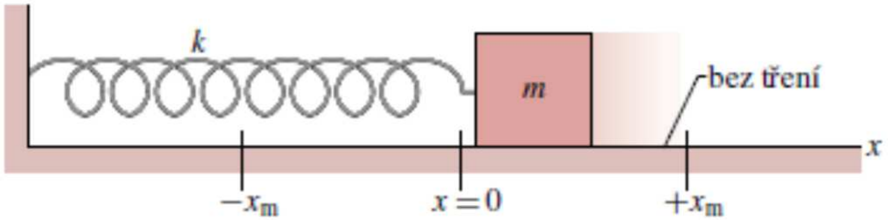
$$m\frac{d^2y}{dt^2} = -ky$$

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0$$

2nd order ordinary differencial equation

Angular frequency $\omega_0^2 = \frac{k}{m}$

Angular frequency and oscillation period



- Angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

- Period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

General solution of equation of motion

General solution $y = A\sin(\omega t + \varphi)$

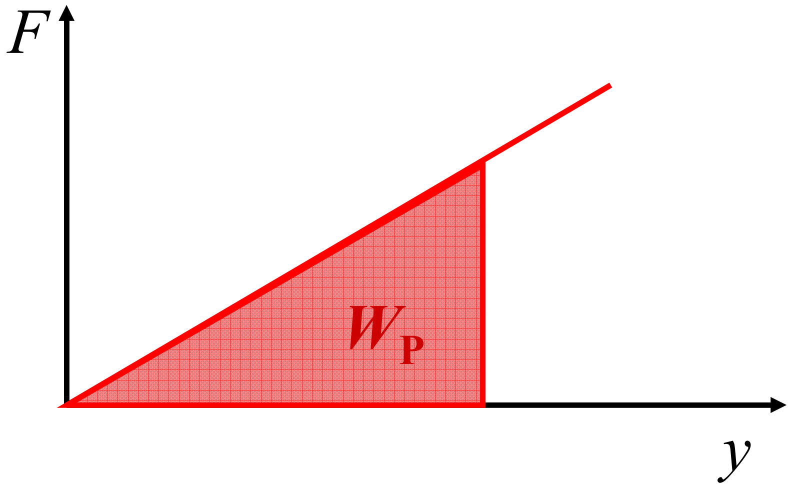
Two parameters – A, φ

Two initial conditions necessary, e.g. maximum displacement and zero velocity

$$y(0) = A \qquad y'(0) = 0 \quad \Rightarrow \quad y = A\sin(\omega t + \frac{\pi}{2}) = A\cos(\omega t)$$

Potential energy of stressed spring

Restoring harmonic force $F = -ky$



$$W_P = -\int_0^{y_0} F dy = \int_0^{y_0} ky dy = \frac{1}{2} k y_0^2$$

Derivation of equation of motion from energy balance

- Kinetic energy of mass

$$W_K = \frac{1}{2} m v^2$$

- Potencial energy of deformed spring

$$W_P = \frac{1}{2} k y^2$$

Energy conservation law – isolated system mass + spring

$$\frac{1}{2} m v^2 + \frac{1}{2} k y^2 = konst.$$

Velocity and acceleration for oscillatory motion

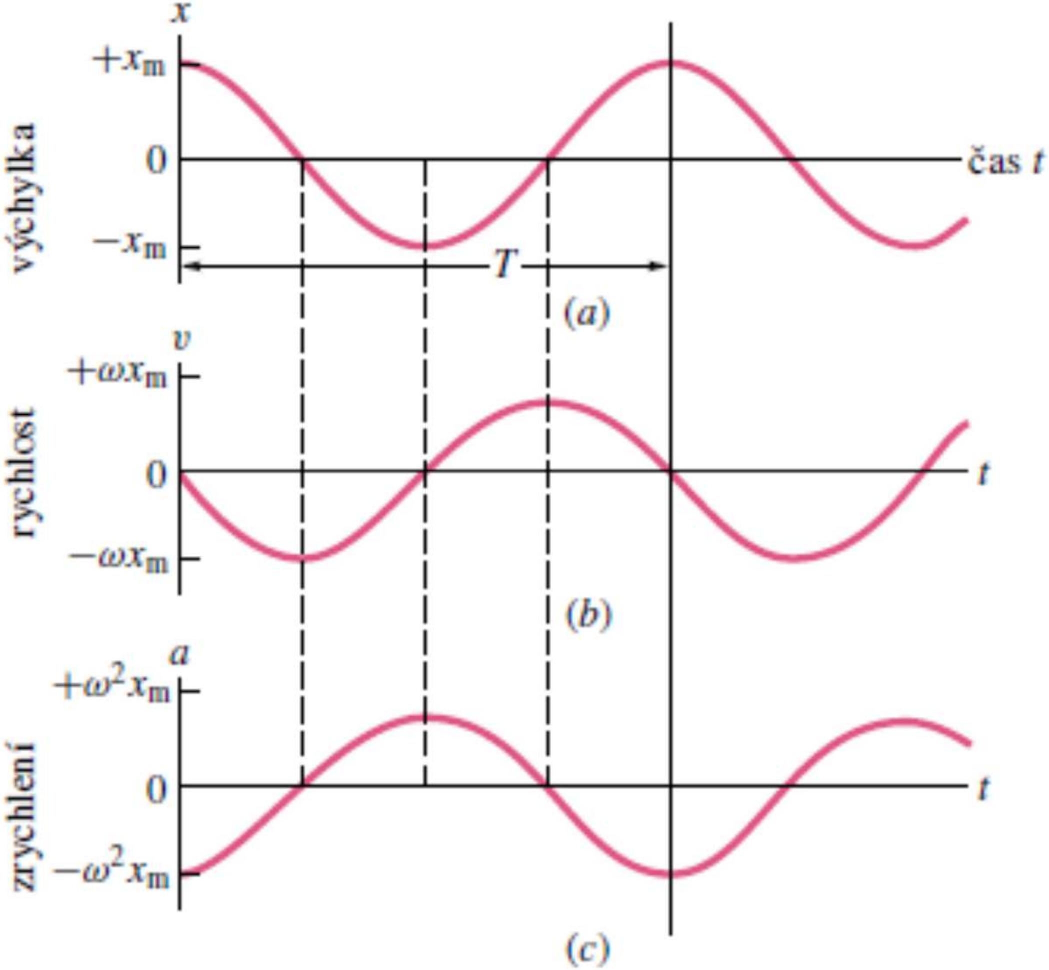
- Displacement $y = A\sin(\omega t + \varphi)$
- Velocity

$$v = \frac{dy}{dt} = A\omega\cos(\omega t + \varphi) = A\omega\sin(\omega t + \varphi + \frac{\pi}{2})$$

- Acceleration $a = -\omega^2 y$

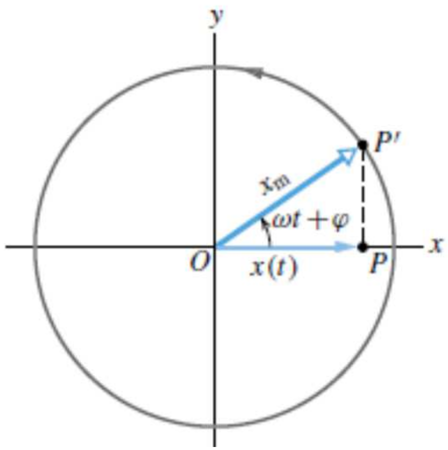
$$a = \frac{dv}{dt} = -A\omega^2\sin(\omega t + \varphi) = A\omega^2\sin(\omega t + \varphi + \pi)$$

Displacement, velocity and acceleration

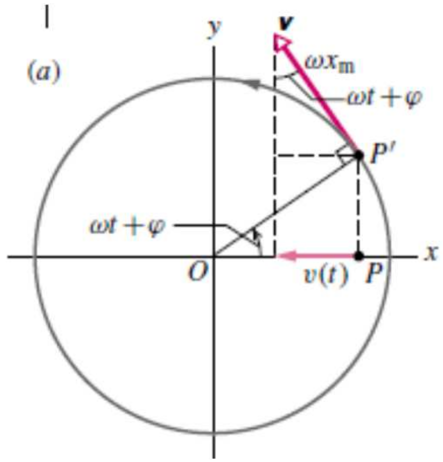


Circular motion analogy

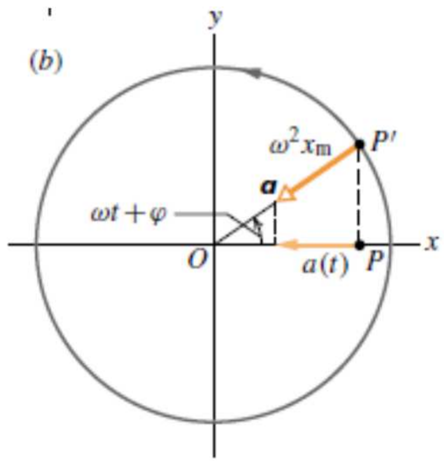
Displacement



Velocity



Acceleration



Energy of oscillation motion

Doubled frequency for energy changes!

- Kinetic

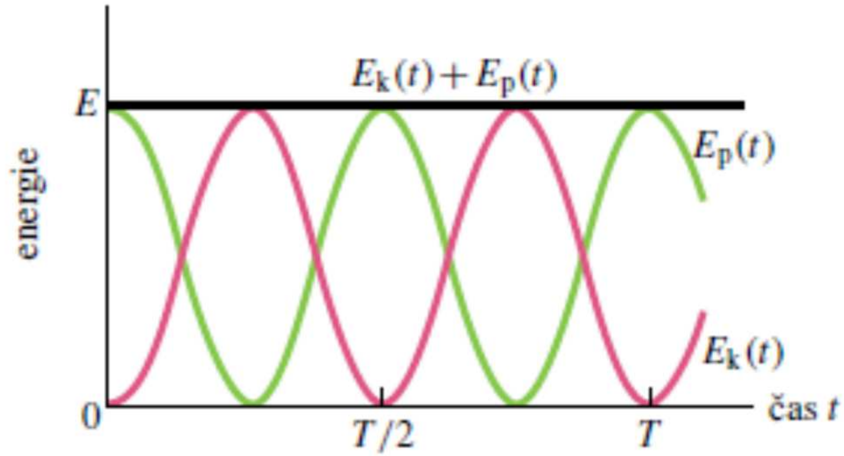
$$W_K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t) = \frac{1}{4}m\omega^2 A^2 (1 - \cos(2\omega t))$$

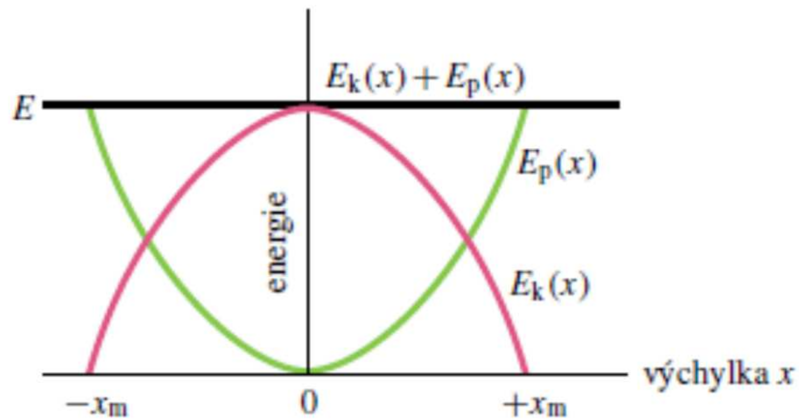
- Potential

$$W_P = \frac{1}{2}ky^2 = \frac{1}{2}kA^2 \cos^2(\omega t) = \frac{1}{4}kA^2 (1 + \cos(2\omega t))$$

Displacement $y = A\cos(\omega t)$

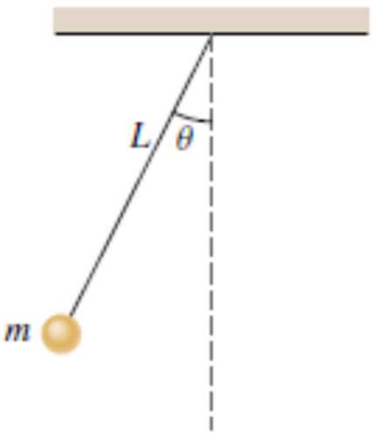
Energy of oscillatory motion

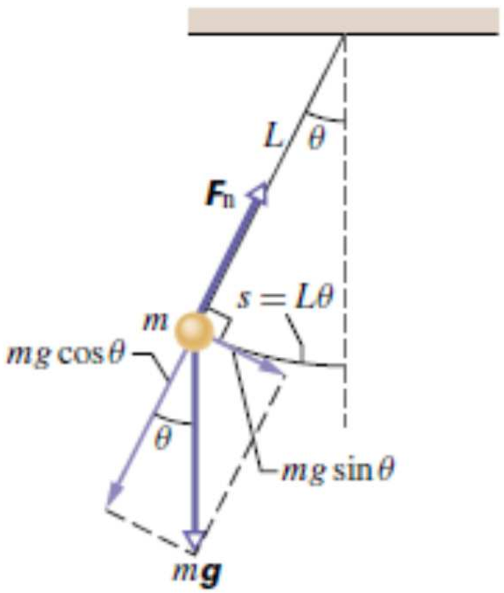




Simple pendulum

Point mass (pendulum bob) and lightweight cord



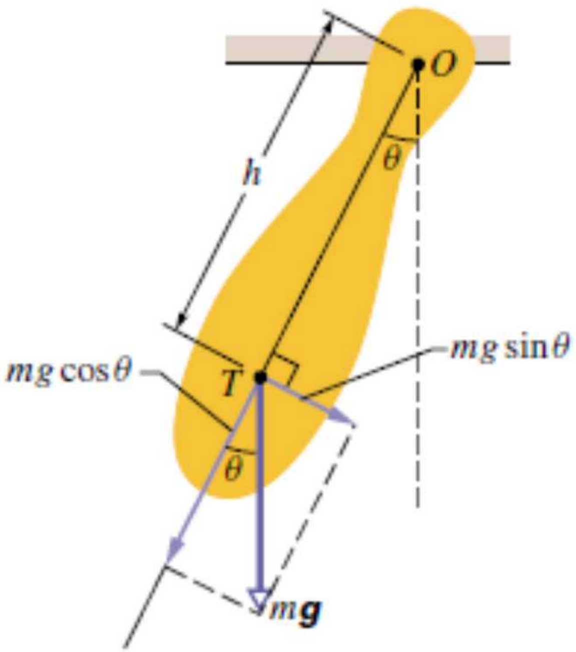


Period

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Physical pendulum

Rigid body rotating by the axis (off center of mass)



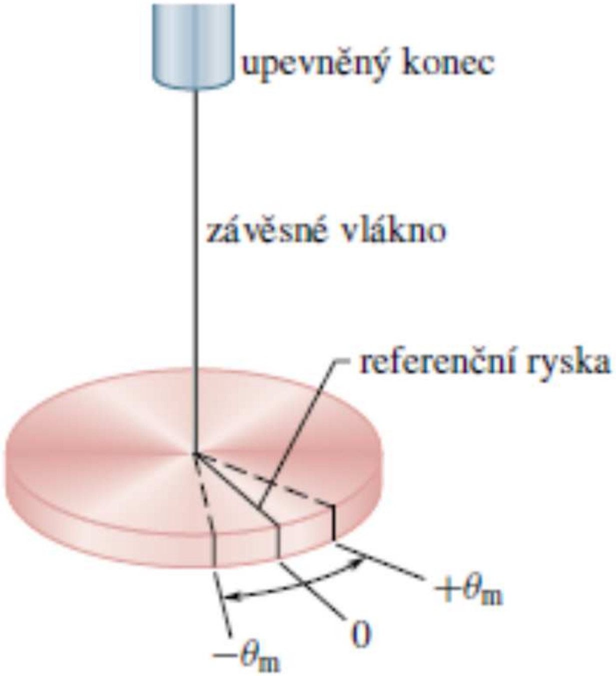
Period

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

inertial moment *I*

Torsion pendulum

Rigid body on lightweight cord deformed by torsion



Period

$$T = 2\pi\sqrt{\frac{I}{\varkappa}}$$

inertial moment I

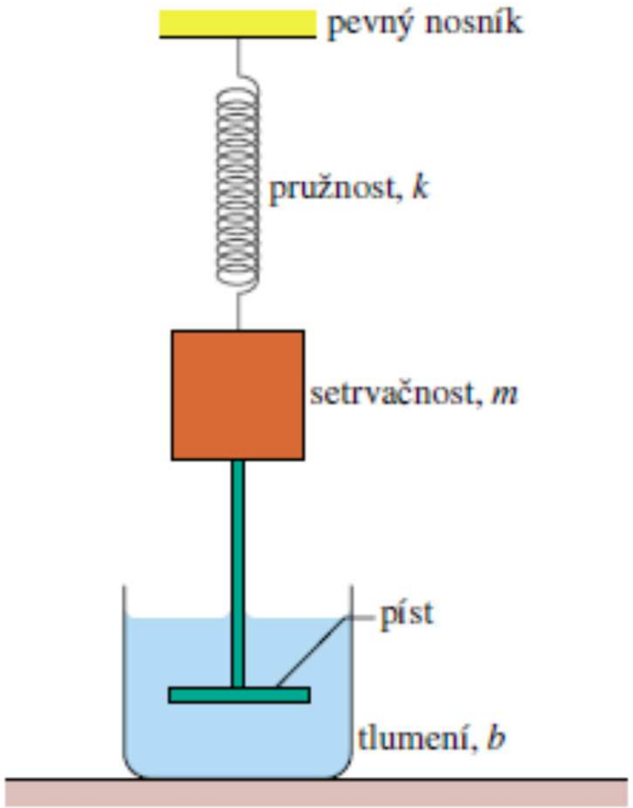
torsional toughness κ

torque (moment of force)

$$M = -\varkappa\theta$$

Damped oscillations

Damping force $F_v = -kx - bv$ Equation of motion



$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$x(t) = x_m e^{-bt/(2m)} \cos(\omega' t + \varphi)$$

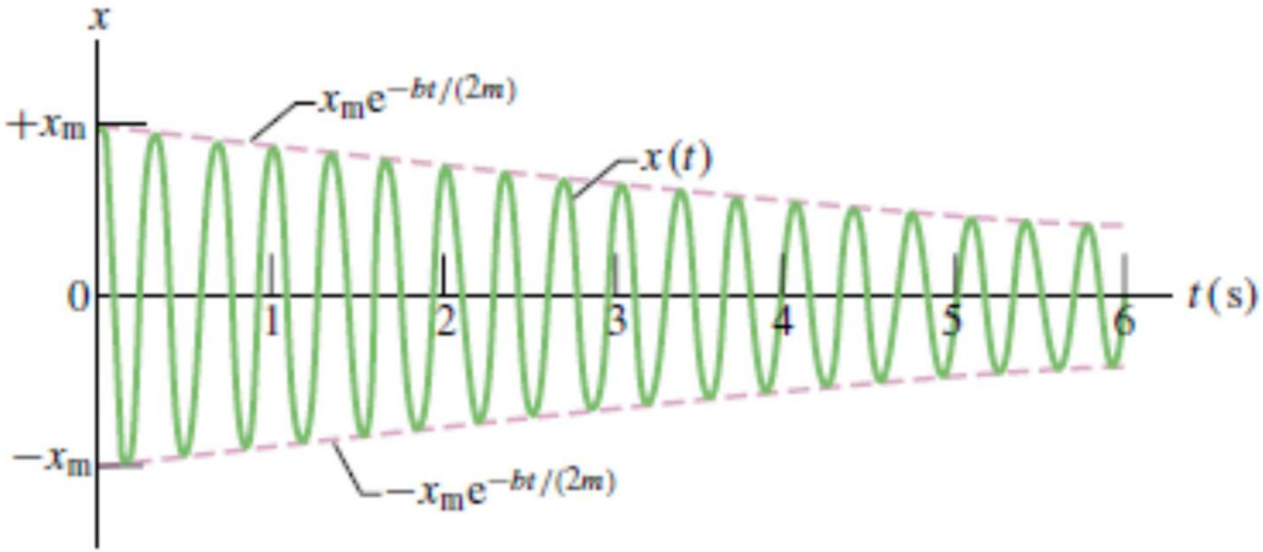
$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

Damped oscillations

exponential amplitude decay

$$x(t) = x_m e^{-bt/(2m)} \cos(\omega' t + \varphi)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$



Attenuation
and logarithmic decrement

Amplitude decays after every period by

Attenuation

$$\frac{x_m e^{-\frac{bt}{2m}}}{x_m e^{-\frac{b(t+T)}{2m}}} = e^{\frac{bT}{2m}}$$

Logarithmic decrement [Bell, dB]

$$\log \frac{x(t)}{x(t+T)} = \frac{bT}{2m}$$

Example

Logarithmic decrement 1dB (0.1B) means decay of oscillation amplitude per cycle

$$\frac{x(t)}{x(t+T)} = 10^{0.1} = 1.259$$

3dB corresponds to the amplitude decay per cycle approx. to one half

$$\frac{x(t)}{x(t+T)} = 10^{0.3} = 1.995$$

Forced vibrations

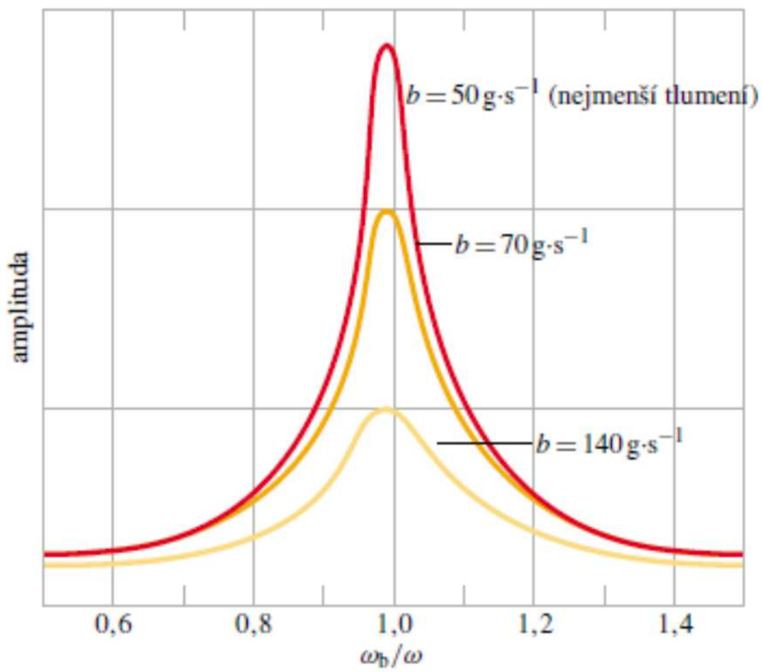
Oscillation is forced by periodic force at the frequency ω_b

$$x(t) = x_m \cos(\omega_b t + \varphi)$$

Oscillation amplitude

depends on frequency

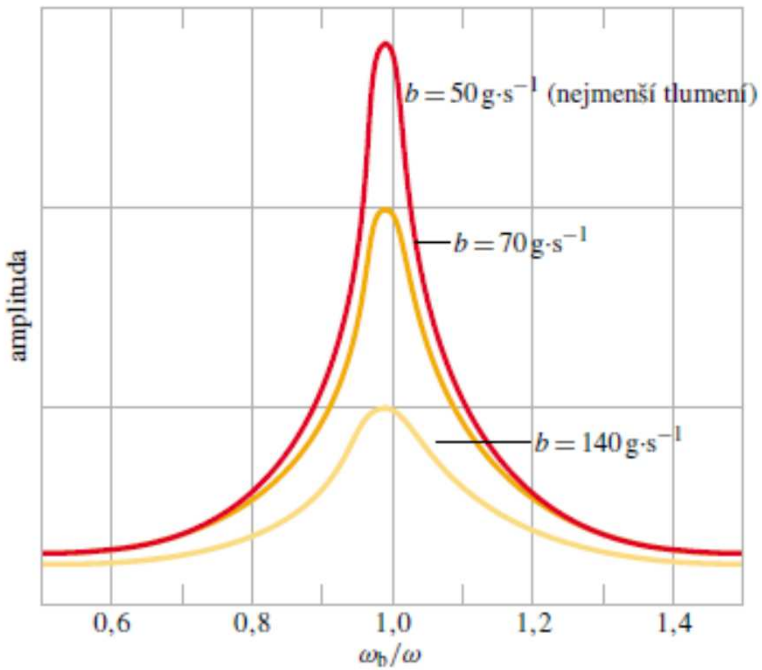
Damping constant



Resonance

Increase of amplitude in the vicinity ω_{b}

$$x(t) = x_m \cos(\omega_b t + \varphi)$$



Resonance frequency

$$\omega_r = \sqrt{\omega_b^2 - \left(\frac{b}{2m}\right)^2}$$

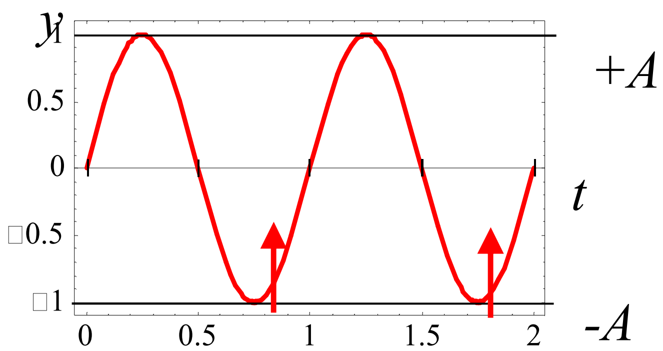
Parametric resonance

Different from resonance, change of oscillation parameters at the specific phase of oscillation

Example:

Swing – change in mass distribution in boundary displacement, keeps displacement amplitude without damping

Joo-joo (Maxwell's pendulum)



Superposition of oscillations with the same frequency

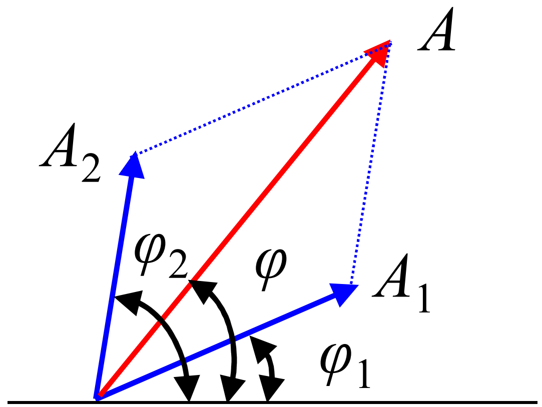
- Calculation

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)$$

$$\tan\varphi = \frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2}$$

- Graphical method

by phase vectors



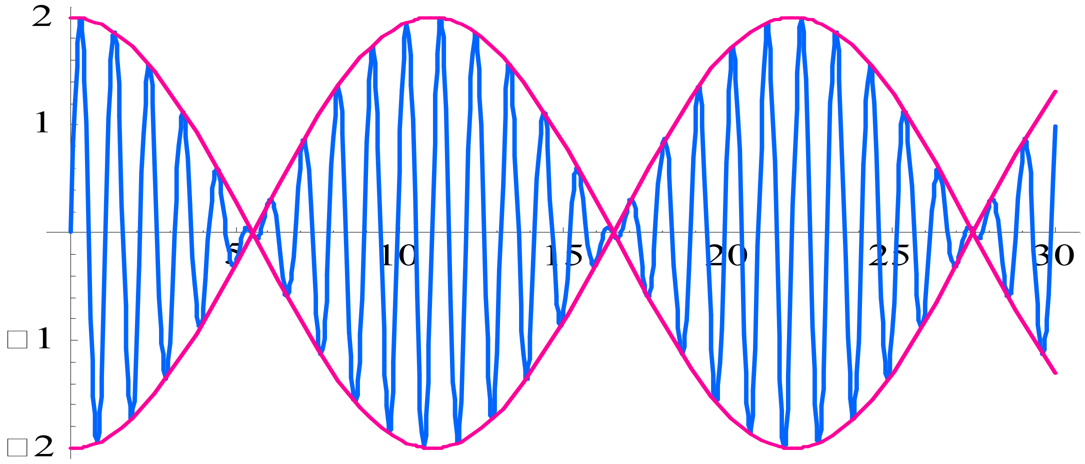
Beats

Oscillations of the same amplitude - superposition

$$y_1 = A\sin(\omega_1 t + \varphi_1) \quad y_2 = A\sin(\omega_2 t + \varphi_2)$$

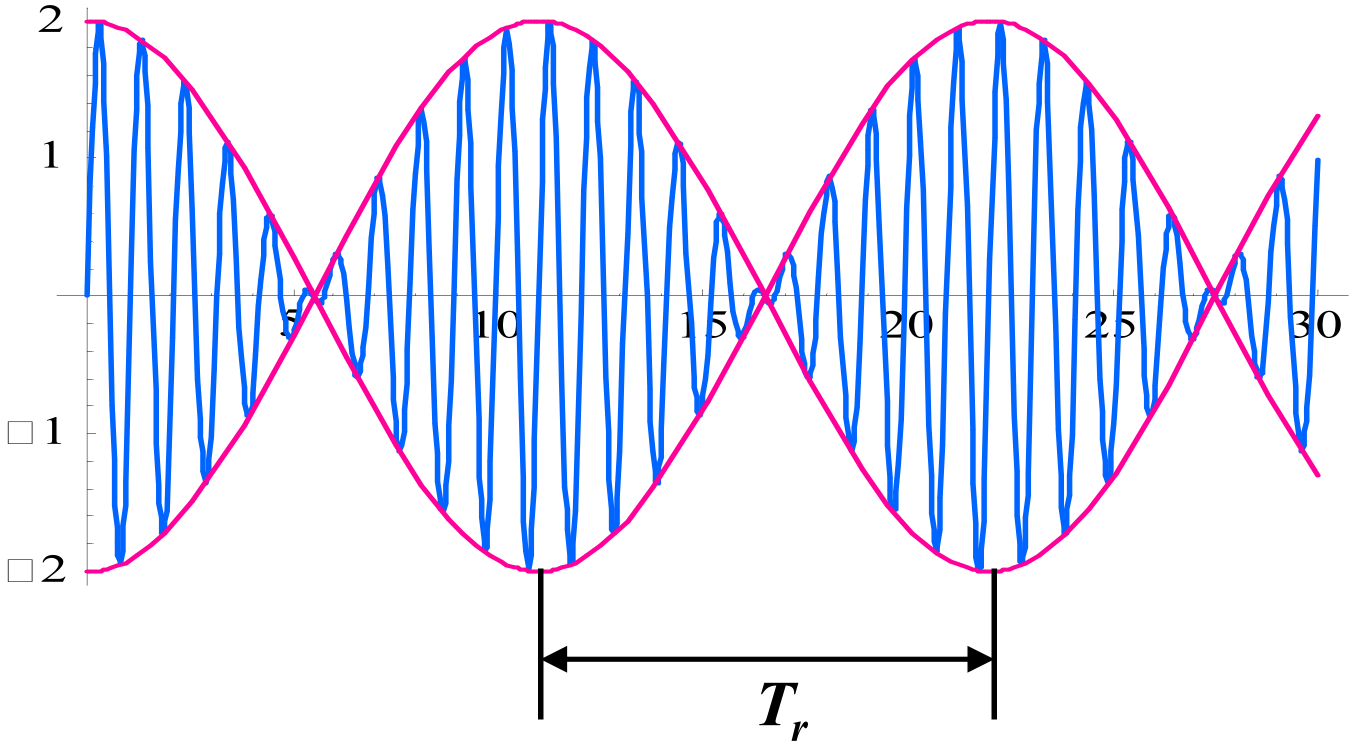
$$y = y_1 + y_2 =$$

$$= 2A\cos[\frac{1}{2}(\omega_1 - \omega_2)t + \frac{1}{2}(\varphi_1 - \varphi_2)]\sin[\frac{1}{2}(\omega_1 + \omega_2)t + \frac{1}{2}(\varphi_1 + \varphi_2)]$$



Beats

Period of beats $f_r = |f_1 - f_2|, \quad T_r = 1/f_r$



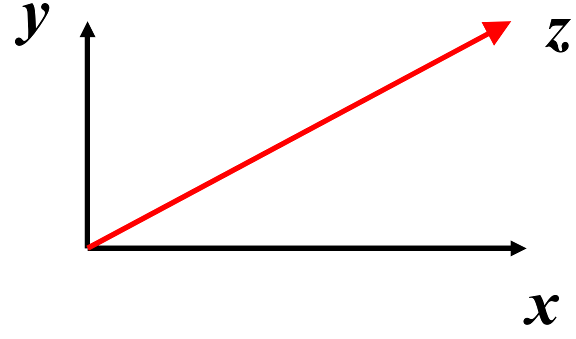
Lissajous pattern

Superposition of oscillations with different direction

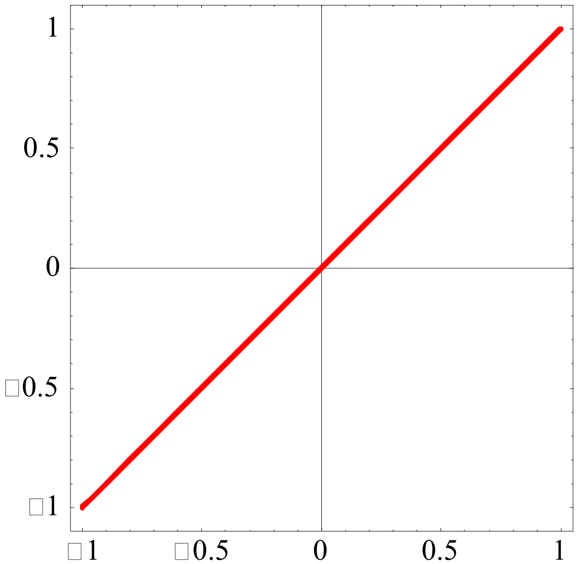
$$x = A_1 \sin(\omega_1 t + \varphi_1), y = A_2 \sin(\omega_2 t + \varphi_2)$$

$$\omega = 2\pi f$$

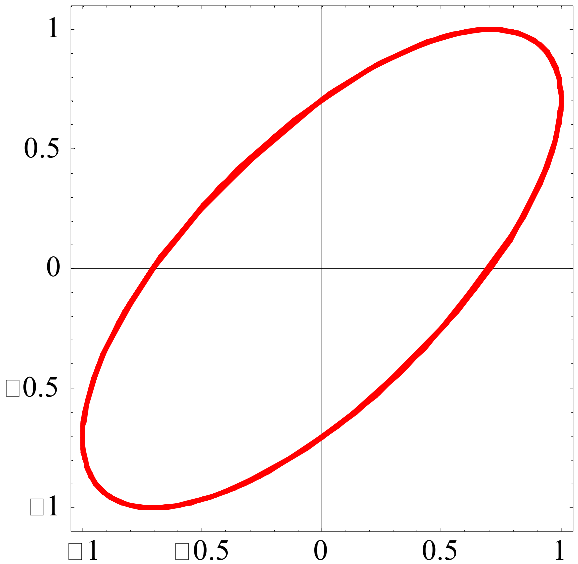
$$z = (x, y)$$



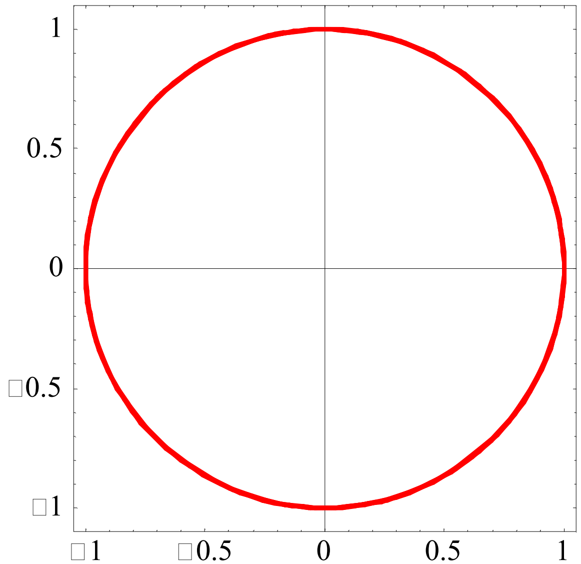
Lissajous pattern



$f_1 : f_2 = 1:1$
$\varphi_1 = 0^o,\ \varphi_2 = 0^o$

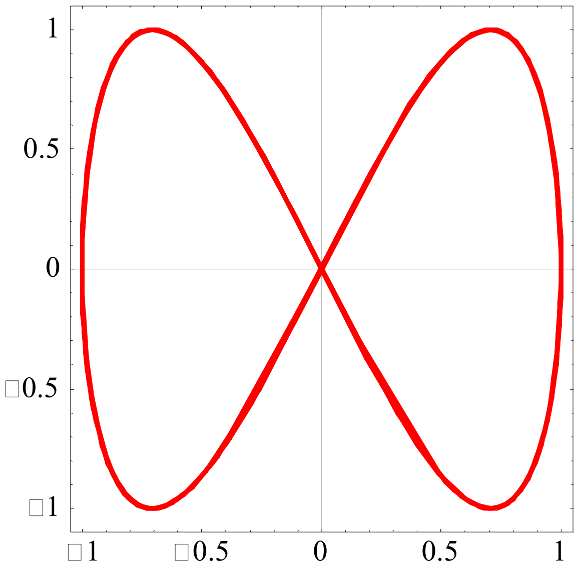


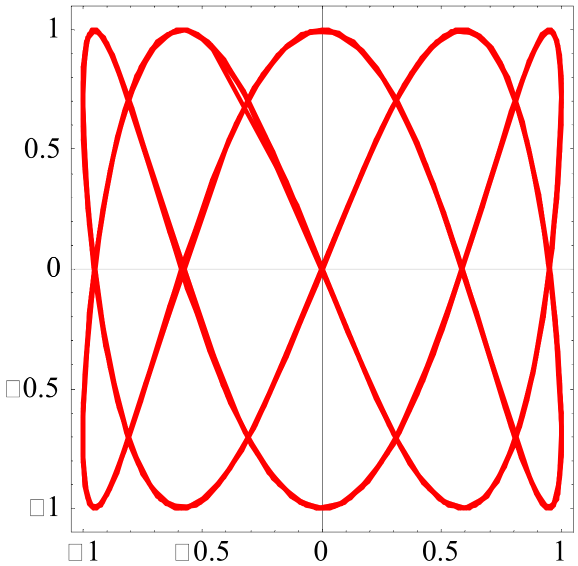
$f_1 : f_2 = 1:1$
$\varphi_1 = 0^o,\ \varphi_2 = 45^o$

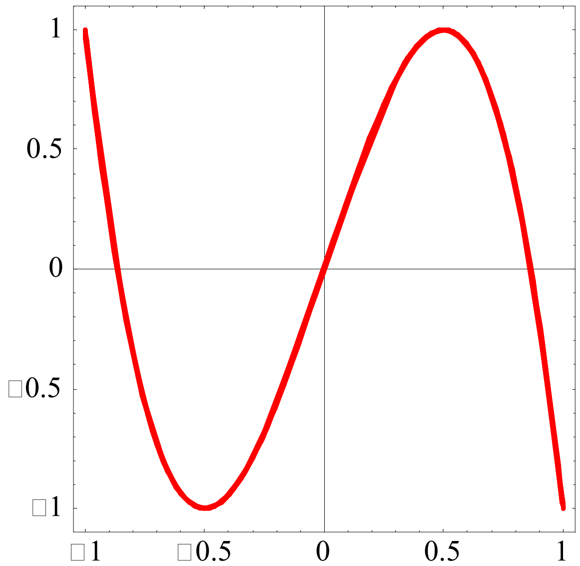


$f_1 : f_2 = 1:1$
$\varphi_1 = 0^o,\ \varphi_2 = 90^o$

Lissajous pattern







$f_1 : f_2 = 1:2$
$\varphi_1 = 0^o$, $\varphi_2 = 0^o$

$f_1 : f_2 = 2:5$
$\varphi_1 = 0^o$, $\varphi_2 = 0^o$

$f_1 : f_2 = 1:3$
$\varphi_1 = 0^o$, $\varphi_2 = 0^o$

Literature

Pictures used from the book:

HALLIDAY, D., R. RESNICK, J. WALKER
Fyzika. Brno: VUTIUM, 2000. díl 2
Mechanika - Termodynamika