Elastic properties of isotropic and anisotropic medium

Continuum deformation, Hook's law, anisotropic medium, stress and strain tensor. Collisions of bodies.

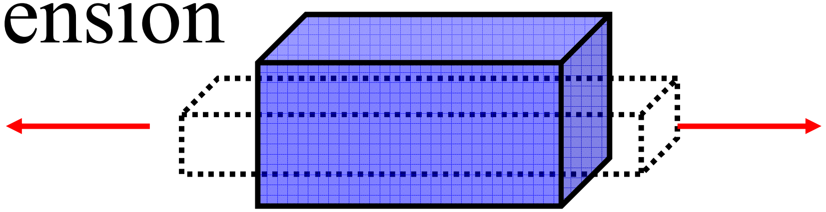
Elastic properties

Continuum – body allowing deformation of size and shape

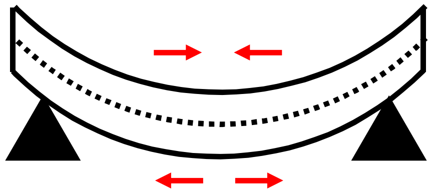
Simple deformations	Composite deformations
Compression, tension	Bending
Shear	Torsion
	…and others

Deformations

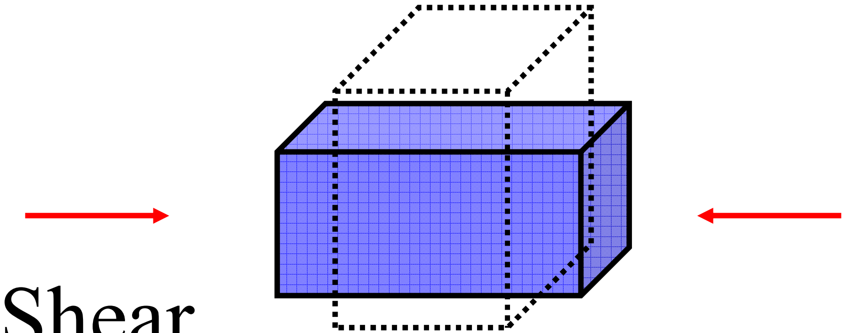
Tension



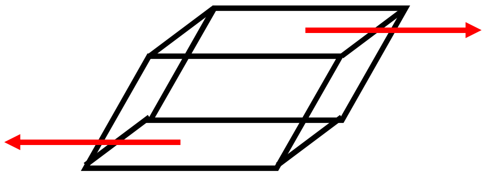
Bending



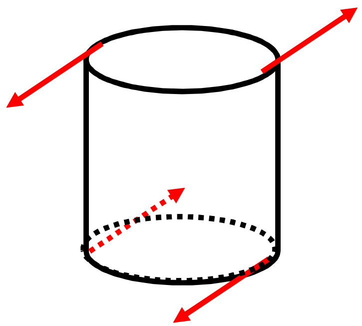
Compression



Shear



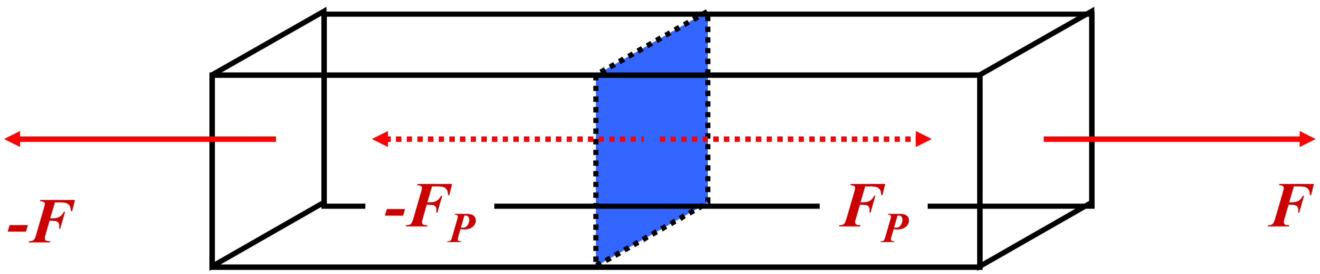
Torsion



Elasticity forces

Normal stress

$$\sigma = \frac{F_P}{S} \quad [Pa]$$



Equilibrium $$F = F_P$$

Elasticity forces

Shear stress

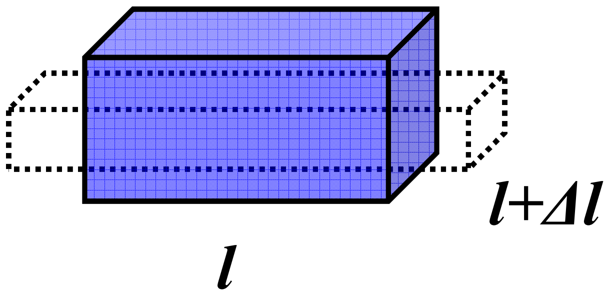
$$\tau = \frac{F_S}{S} \quad [Pa]$$

Strain

Relative deformation/strain - uniaxial

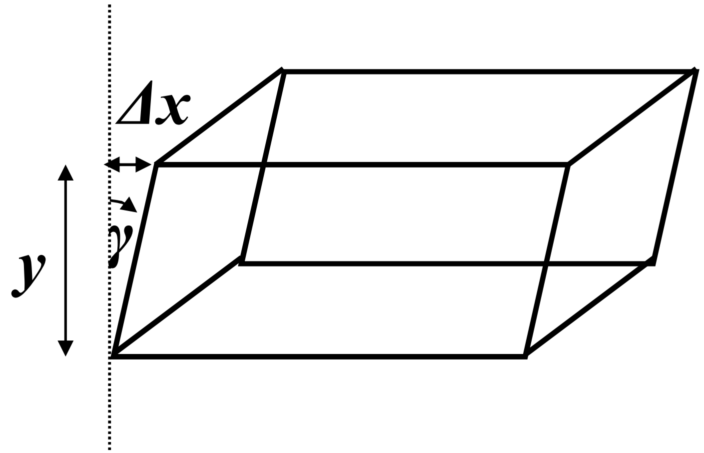
Longitudinal

$$\varepsilon = \frac{\Delta l}{l} \quad [\%]$$



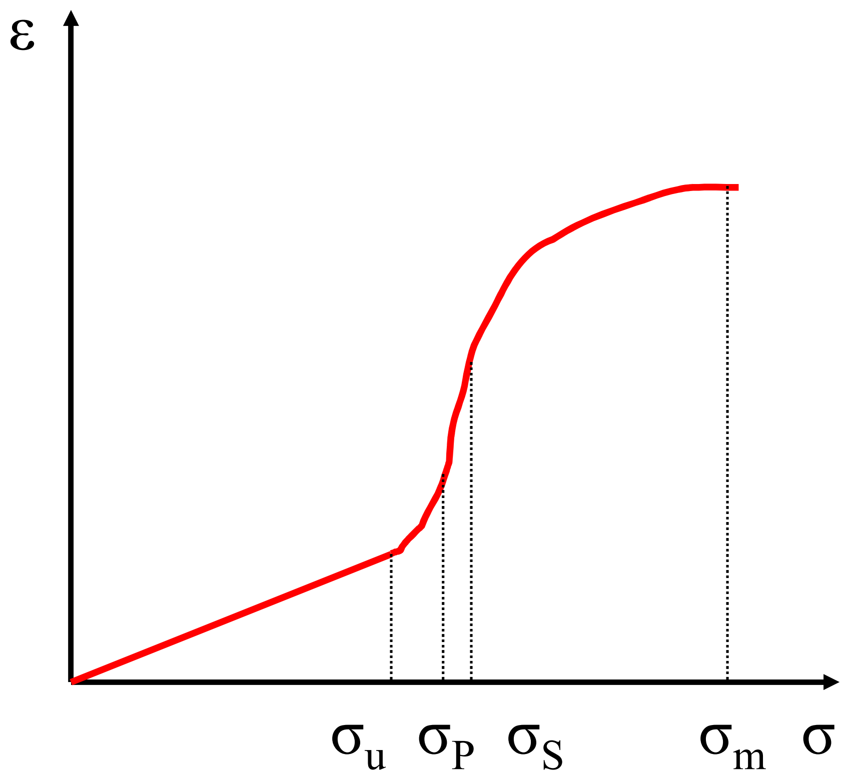
shear

$$\gamma \approx tg(\gamma) = \frac{\Delta x}{y} \quad [\%]$$



Stress-strain relationship

Stress-strain curve



Limits

proportionality σ_u

elasticity σ_P

plasticity σ_S

strength

$$\sigma_m \approx 1 - 100 \cdot 10^7\ Pa$$

Hook's law

Tensile strain

$$\varepsilon = \frac{1}{E}\sigma$$

Shear strain

$$\gamma = \frac{1}{G}\tau$$

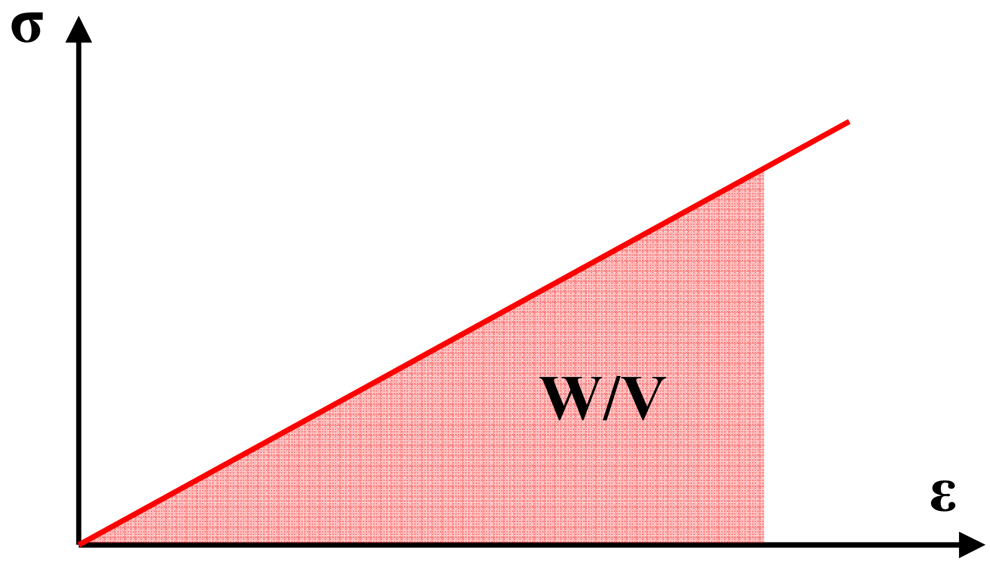
Elastic modulus [Pa]

- Young's modulus - *E*
- Shear modulus - *G*

Energy of elastic deformation

Potential energy of elasticity forces

$$W = \int F dl = V \int \frac{F}{S} \frac{dl}{l} = V \int \sigma d\varepsilon = V \int E \varepsilon d\varepsilon = \left[\frac{1}{2} V E \varepsilon^2 \right] = \left[\frac{1}{2} V \sigma \varepsilon \right]$$



Elastic moduli

Transversal deformation

$$\varepsilon_L = \frac{1}{E}\sigma \qquad \varepsilon_T = -\frac{\nu}{E}\sigma$$

Poisson's ratio

$$\nu = -\frac{\varepsilon_T}{\varepsilon_L}, \qquad 0 < \nu \leq \frac{1}{2}$$

Isotropic medium $$G = \frac{E}{2(1+\nu)}$$

Elastic moduli

	$E [10^{10}\text{Pa}]$	$G [10^{10}\text{Pa}]$	ν [1]
Fe	21	8	0.29
C, fibers	112	52	0.1

3D deformation

Stress in 3 dimensions

$$\varepsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_3$$

$$\varepsilon_2 = -\frac{\nu}{E}\sigma_1 + \frac{1}{E}\sigma_2 - \frac{\nu}{E}\sigma_3$$

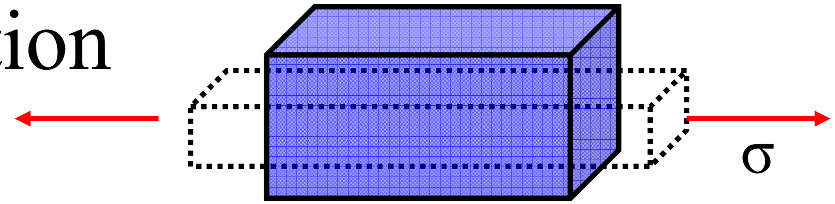
$$\varepsilon_3 = -\frac{\nu}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 + \frac{1}{E}\sigma_3$$

Volume change for uniaxial tension/compression

Parallelepiped deformed by stress σ

Volume before deformation

$$V_0 = abc$$



Volume after deformation

$$V = a\left(1+\frac{1}{E}\sigma\right)b\left(1-\frac{\nu}{E}\sigma\right)c\left(1-\frac{\nu}{E}\sigma\right) \approx V_0 + V_0(1-2\nu)\frac{1}{E}\sigma$$

$$\frac{V-V_0}{V_0} = (1-2\nu)\frac{1}{E}\sigma$$

Isotropic stress

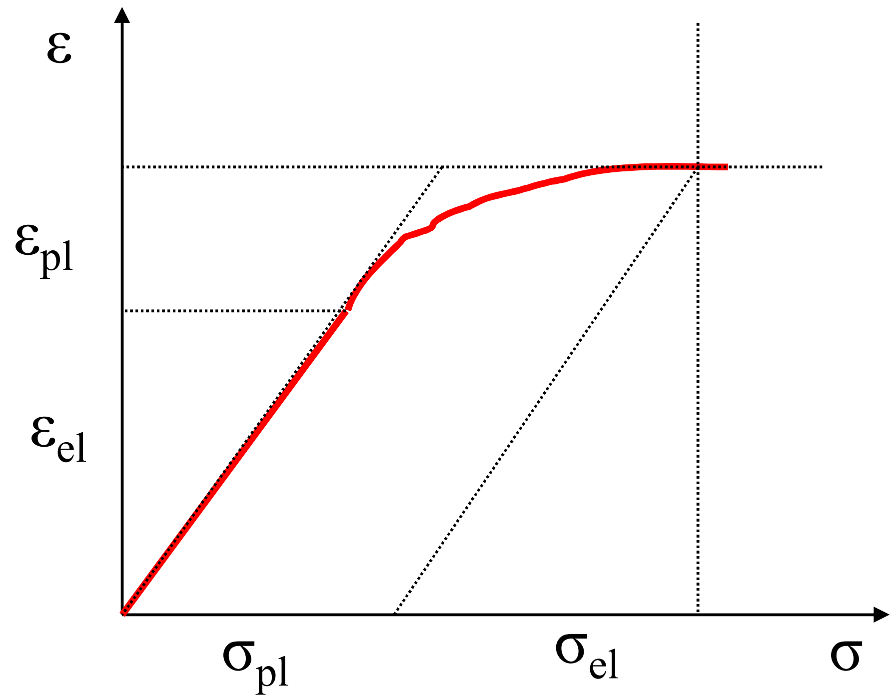
Isotropic stress in all directions (e.g. liquids, hydrostatic pressure)

$$\sigma_1 = \sigma_2 = \sigma_3 = -p$$

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{E}(1-2\nu)(-p)$$

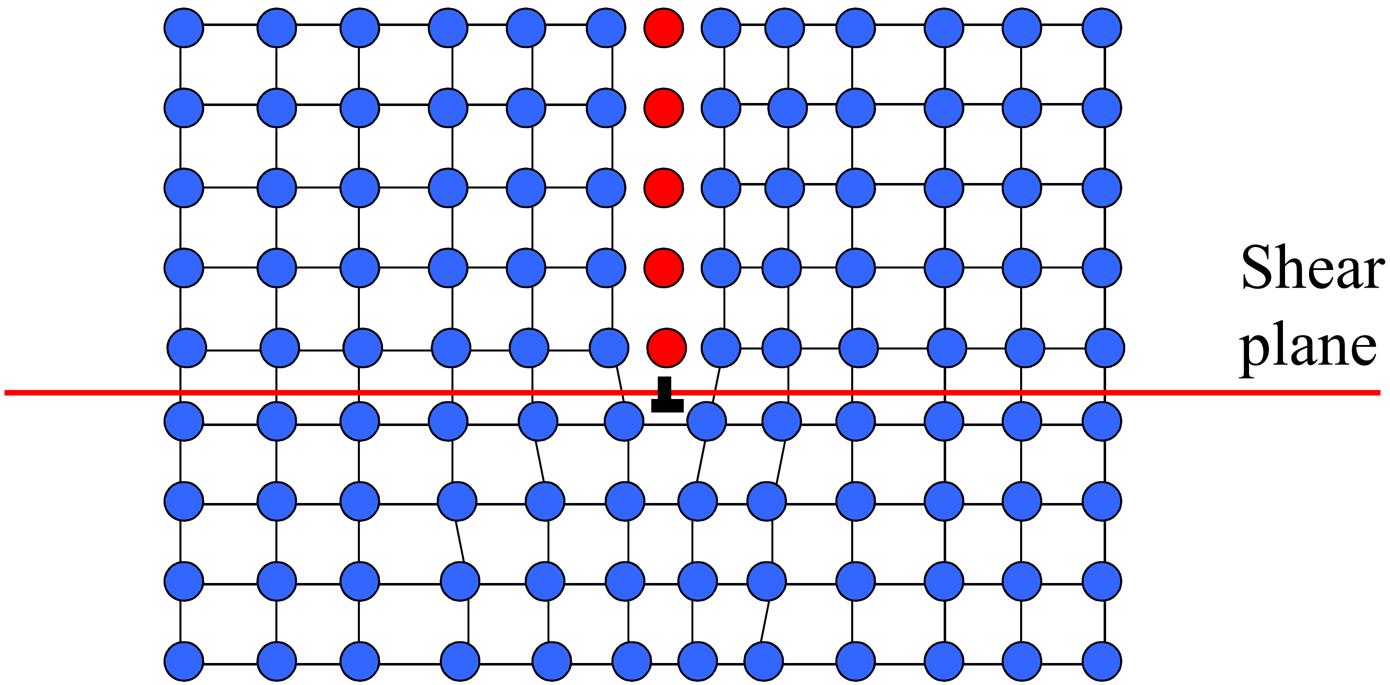
Elastic vs. plastic deformation

- Elastic – reversible
- Plastic – irreversible



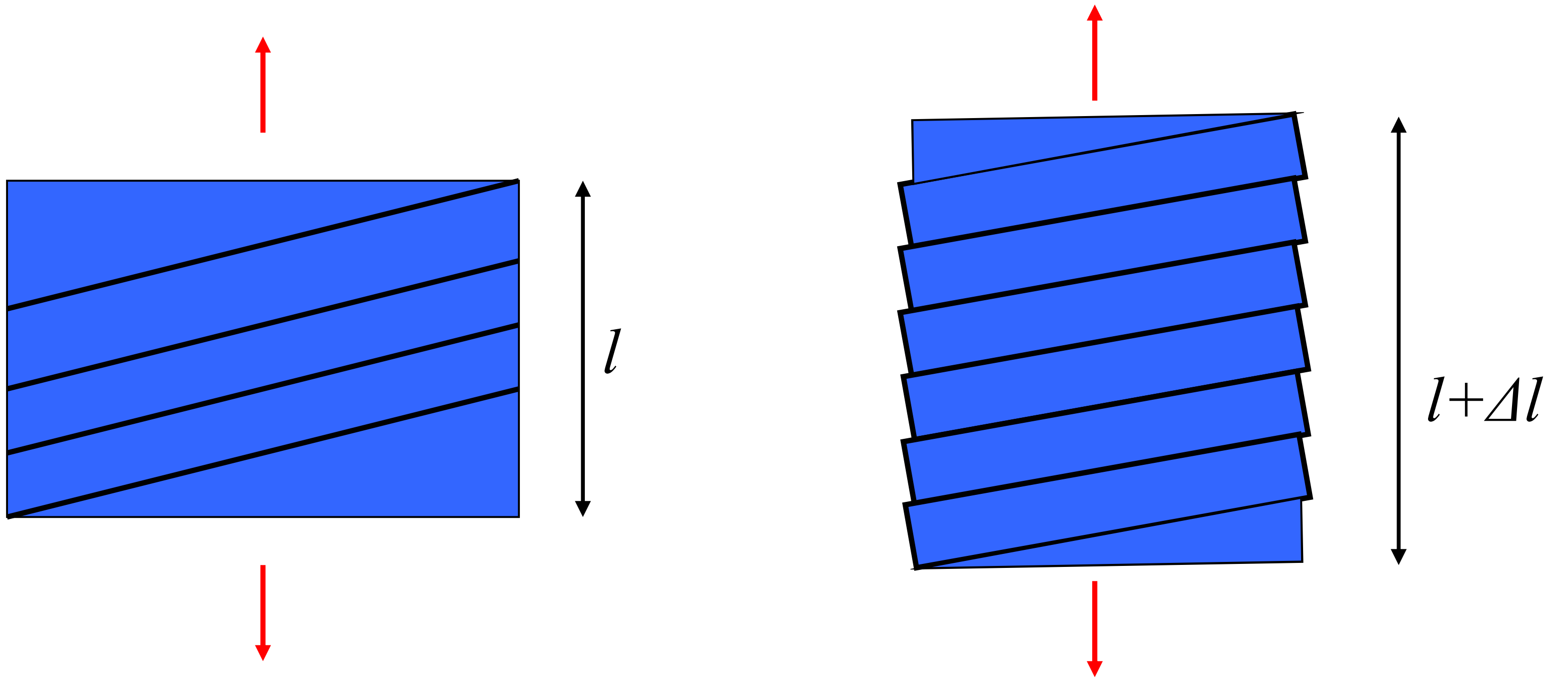
Dislocation

- Edge
- Screw



Plastic deformation

Irreversible dislocation movement



Anisotropic medium

Properties variable in different directions

Materials:

- Crystals
- Composites (e.g. reinforced by fibers - laminates, wood,…)
- Plastic foils, metal sheets, paper (direction of foil stretching, rolling, …)
- Ceramics (pressing direction)

Anisotropic stress and strain

Strain

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix}$$

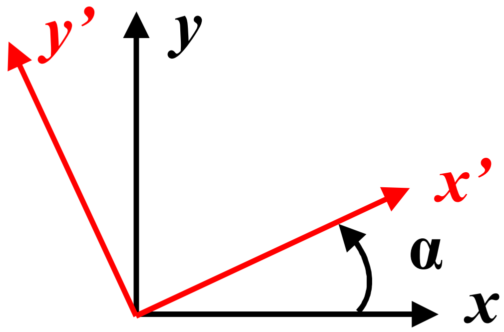
stress

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

2nd order tensors

Tensor coordinates transformation

Coordinates in rotated coordinate system



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varepsilon_{ij}^{'} = \sum_{k=1}^{3} \sum_{l=1}^{3} a_{ik} a_{jl} \varepsilon_{kl}$$

Example of coordinate transformation

Uniaxial stress

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \sigma'_{11} & \sigma'_{12} & 0 \\ \sigma'_{12} & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma'_{11} = a_{11}a_{11}\sigma = \sigma\cos^2\alpha$$

$$\sigma'_{12} = a_{11}a_{21}\sigma = -\sigma\sin\alpha\cos\alpha$$

$$\sigma'_{13} = a_{11}a_{31}\sigma = 0$$

$$\sigma'_{22} = a_{21}a_{21}\sigma = \sigma\sin^2\alpha$$

$$\sigma'_{23} = a_{21}a_{31}\sigma = 0$$

$$\sigma'_{33} = a_{31}a_{31}\sigma = 0$$

Hook's law in anisotropic medium

Elastic moduli are 4th rank tensors

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl}\varepsilon_{kl} \qquad \varepsilon_{ij} = \sum_{k,l=1}^{3} s_{ijkl}\sigma_{kl}$$

Elastic modulus is symmetrical – reduction of number of independent moduli coordinates to 21 (isotropic medium - 2 independent moduli)

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$

Elastic moduli

Tensorial vs. matrix representation

$$c_{\alpha\beta} = c_{ijkl} \quad \alpha, \beta = 1,2,\ldots,6$$

$$s_{\alpha\beta} = \begin{cases} s_{ijkl} & \alpha, \beta = 1,2,3 \\ 2s_{ijkl} & \alpha = 1,2,3; \beta = 4,5,6 \\ 4s_{ijkl} & \alpha, \beta = 4,5,6 \end{cases}$$

Planar orthotropic medium

Example: Different moduli in two perpendicular directions

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \qquad \begin{pmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{13} & s_{23} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{pmatrix}$$

Transformation of elastic moduli

4th rank tensor

$$C'_{ijkl} = \sum_{m,n,p,q=1}^{3} a_{im} a_{jn} a_{kp} a_{lq} C_{mnpq}$$

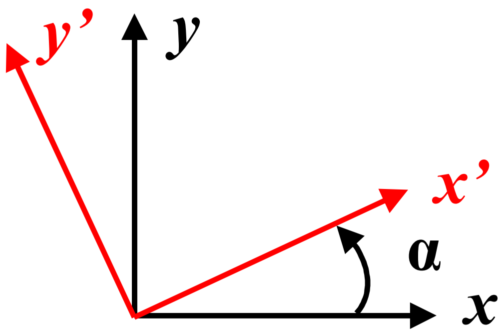
Example: Elastic modulus value in x-y plane for orthotropic medium

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C'_{1111} = C_{1111}\cos^4\alpha + 2C_{1122}\sin^2\alpha\cos^2\alpha + C_{2222}\sin^4\alpha$$

Planar orthotropic medium

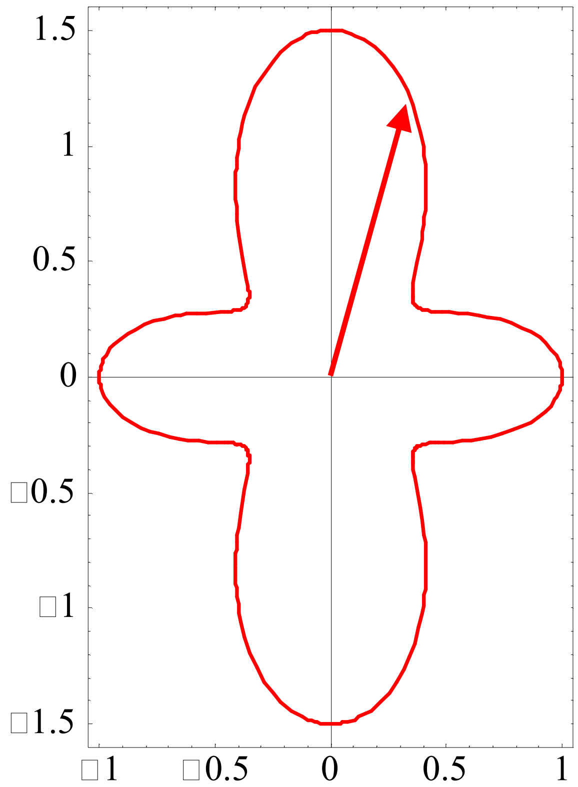
$$C'_{1111} = C_{1111}\cos^4\alpha + 2C_{1122}\sin^2\alpha\cos^2\alpha + C_{2222}\sin^4\alpha$$



Př. C_{1111}=E,

C_{2222}=1.5E,

C_{1122}=-1/4E



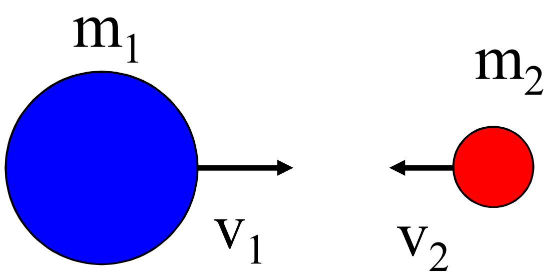
Collisions of bodies

Loss of energy ΔW

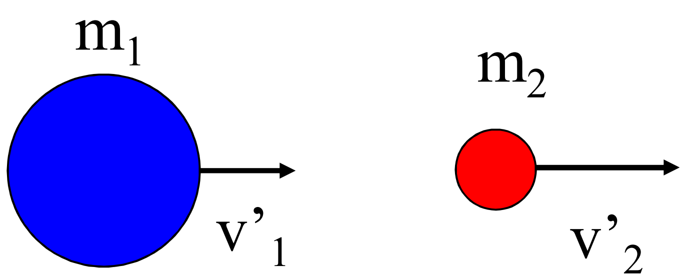
- Elastic collision $\Delta W = 0$
- Inelastic collision $\Delta W_{\max}$
- General $0 < \Delta W < \Delta W_{\max}$

Elastic collision

Before



after



$$m_1 v_1 - m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

Elastic collision

Body velocities after collision

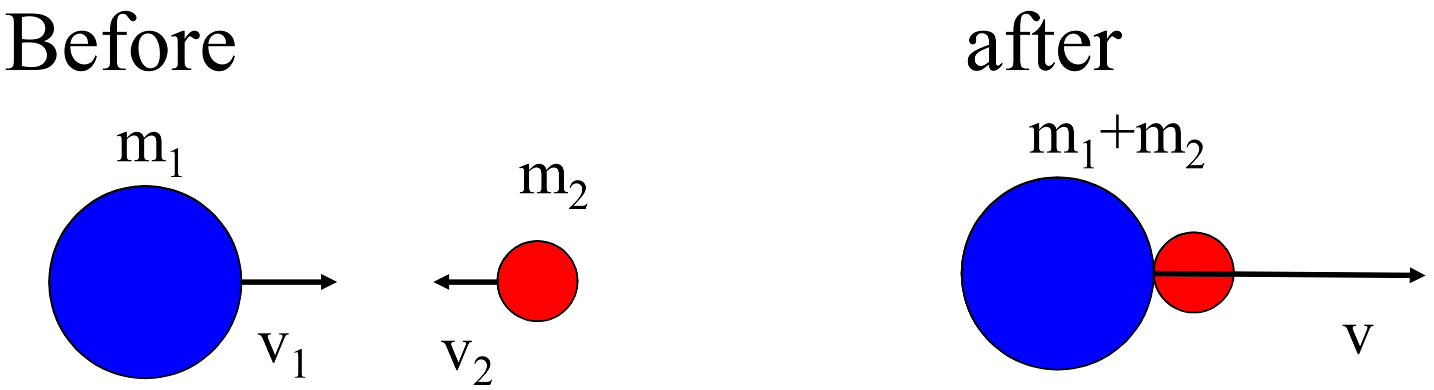
$$v_1' = \frac{(m_1 - m_2)v_1 - 2m_2 v_2}{m_1 + m_2}$$

$$v_2' = \frac{(m_1 - m_2)v_2 + 2m_1 v_1}{m_1 + m_2}$$

Special case

$$m_1 = m_2 = m, v_2 = 0 \Rightarrow v_1' = 0, v_2' = v_1$$

Inelastic collision



$$m_1 v_1 - m_2 v_2 = (m_1 + m_2)v$$

$$v = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2}$$

Inelastic collision

Loss of energy

$$\Delta W = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_1 v_2^2 - \frac{1}{2}(m_1 + m_2) v^2 =$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1 + v_2)^2 \geq 0$$

Literature

Pictures were used from the book:

HALLIDAY, D., R. RESNICK, J. WALKER
Fyzika. Brno: VUTIUM, 2000. díl 2
Mechanika - Termodynamika