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YOUTH AND SPORTS

Preparation of the international Ph.D. study programme "Environmental Engineering" CZ.02.2.69/0.0/0.0/16_018/0002660

Transport processes in rock and soil

Lecture 2

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Reminder

- Porous medium
- Homogenization
- Representative elementary volume (REV)
- Porosity
- Darcy’s Law
  - Hydraulic conductivity
  - Piezometric head
- Average pore velocity X Darcy velocity

Plans

- Darcy’s Law in 3D – differential form
- Full system of flow equations (+balance)
- Characterisation of the hydraulic conductivity K
- Boundary conditions (for diff. eq.)
  - Cases for groundwater flow configurations

Water flow velocity – reminder

$$\boldsymbol{v} = \frac{1}{V^w_{REV}} \int\limits_{V^w_{REV}} \boldsymbol{v}^{(mic)} \mathrm{d}V^w$$

Water/pore volume

v … Particle movement from point to point

$$\boldsymbol{q} = \frac{1}{V_{REV}} \int\limits_{V_{REV}} \boldsymbol{v}^{(mic)} \mathrm{d}V$$

Total volume

q … Amount of water (across unit area]

$$\frac{Q}{S} = |\vec{q}|$$

$$q\,S = v\,S\,n$$

$$q = v\,n$$

$$v = \frac{q}{n}$$

$$v > q$$

q … "Darcy velocity" (flow rate density)

v … (average) pore velocity

Differential form

- Experimental column … infinitesimally small distance

- Spatial coordinates
- Generalized Darcy’s Law

$h(\vec{x})$

$p(\vec{x})$

$\vec{q}(\vec{x})$

K

3D

$$\vec{q} = -\overline{\overline{K}}\,\nabla h$$

ZOBECNĚNÝ DARCY Z.

$$\vec{q} = -K\left(\frac{\nabla p}{\rho g} + \nabla z\right)$$

$$q_x = -K\frac{\partial h}{\partial x} \quad \ldots$$

Hydraulic conductivity

- Controlled by
  - Porous medium properties (microstructure geometry)
  - Fluid properties (viscosity)

$$\vec{q} = -K\left(\frac{\nabla p}{\rho g} + \nabla z\right)$$

$$\vec{q} = -\frac{k}{\mu}\left(\nabla p + \rho g \nabla z\right)$$

K ← VLASNOST PROSTŘEDÍ

– || – TEKUTINY ... VISKOZITA

$$K = \frac{k \cdot \rho \cdot g}{\mu} = \frac{k \cdot g}{\nu}$$

[m/s]

ν ... KINEMAT. VISK [m^2/s] — Kinematic viscosity

μ ... DYNAM. VISK [Pa·s] — Dynamic viscosity

PERMEABILITA

k ... PROPUSTNOST (PROSTŘEDÍ) [m^2] — Permeability

$1D \approx 10^{-12} m^2$

DARCY

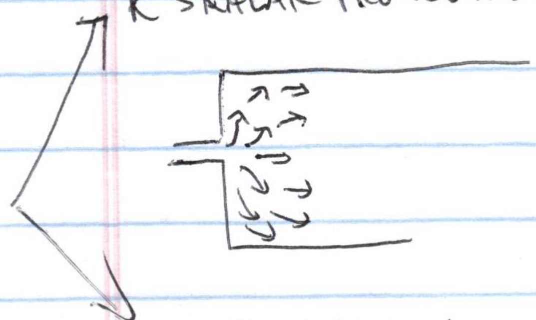
Special unit 1 “Darcy”

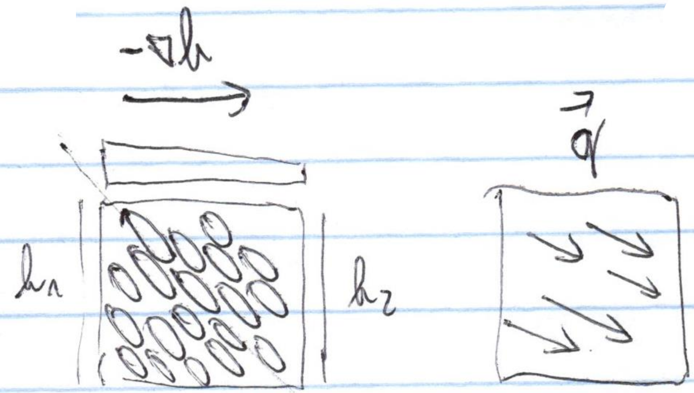
K / k … factor about 10^7 for water

Pore geometry … anisotropy … K

Scalar for isotropic medium

K SKALÁR PRO IZOTROPNÍ





ANIZOTROPIE

Symmetric tensor for anisotropic … 6 components

K … TENZOR (SYMETR.) 6 SLOŽEK

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} \\ \cdot & K_{22} & \cdot \\ \cdot & \cdot & K_{33} \end{pmatrix} \qquad \begin{pmatrix} K_{xx} & K_{xy} & \cdots \\ \vdots & \ddots & \\ & & \ddots \end{pmatrix}$$

TYPICKY: ORTOTROPNÍ

Common case is "orthotropic"

$$\begin{pmatrix} K_x & & 0 \\ & K_y & \\ 0 & & K_z \end{pmatrix}$$

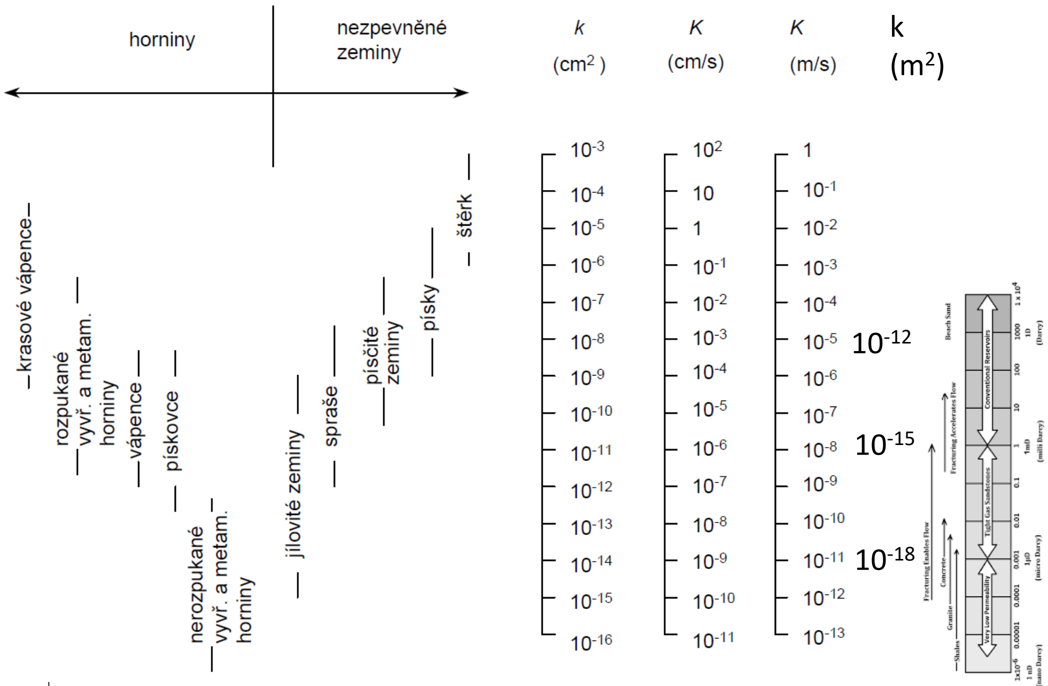
(SOUŘ. SYSTÉM V HLAVNÍCH SMĚRECH)

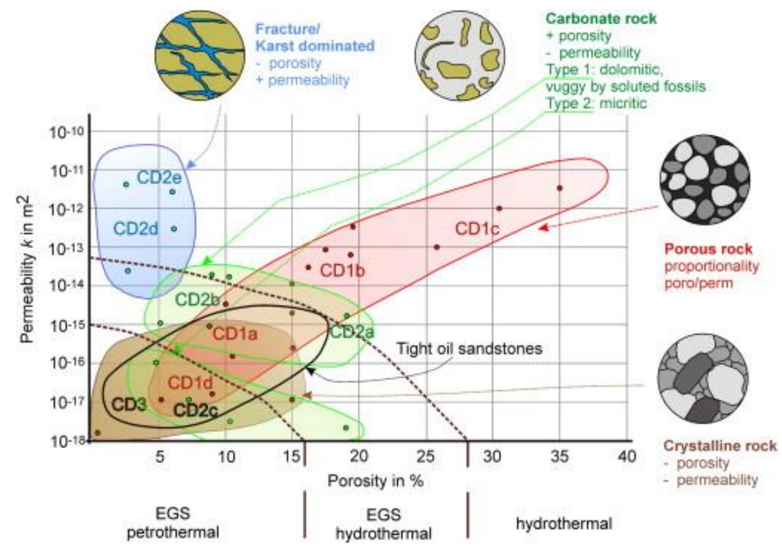
Coordinate system in principal directions

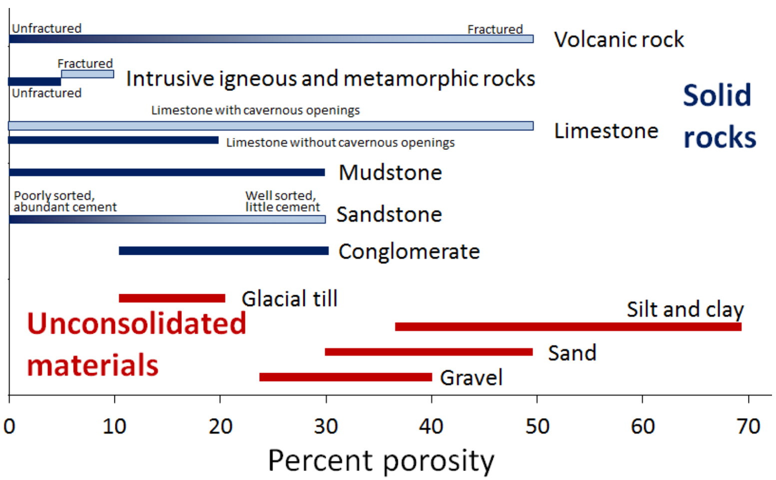
PODZEMNÍ VODA V SEDIMENTECH $K_{HORIZ} \gg K_{VERTIK}$

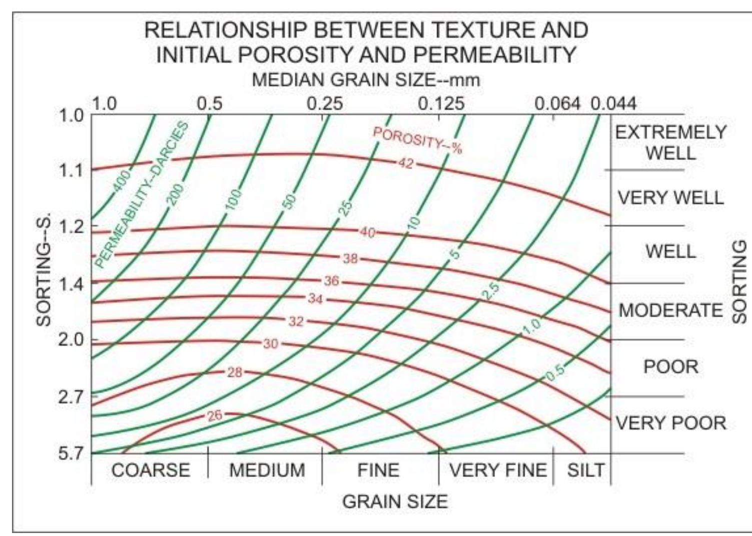
Groundwater in sediments

Properties of rock/soil









Full system of governing equation for porous media flow

SYSTÉM ROVNIC

quantities

VELIČINY $\vec{q}(\vec{x},t)$

$h(\vec{x},t)$

Darcy's Law

- DARCY Z. ✓

Balance equation (mass conservation principle)

- ROVNICE BILANCE

(ZÁK. ZACH. HMOTY, ROVNICE KONTINUITY)

Fully saturated pores

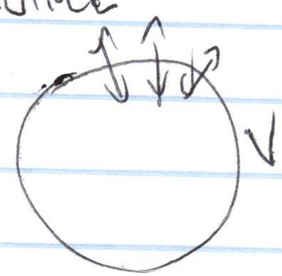
→ PLNĚ NASYCENÉ (SATUROVAT)

→ ČÁSTEČNĚ NASYC.

(PÓRY: VODA + ~~VZDUCH~~)

Partially saturated pores

BILANCE



ZMĚNA V OBJEMU ~ TOK PŘES STĚNU (HRANICI)

Balance: change inside the volume vs flux across the boundary

$$\underbrace{\frac{d}{dt}\int_V \rho \cdot n \, dV}_{\text{HMOTNOST}} = -\int_{\partial V} \rho \vec{q} \, dS + \underbrace{\int_V P \rho \, dV}_{\text{ZDROJE/PROPADY}}$$

mass

↓

OBJEMOVÝ (GAUSS. V.)

Transform to volume integral

Sources/sinks

Volumetric source

P objemový zdroj

$[m^3/m^3/s]$

$$\frac{d}{dt}\underbrace{\int_V \rho \cdot n \, dV}_{\text{HMOTNOST}} = -\int_{\partial V} \rho \vec{q} \, dS + \underbrace{\int_V P \rho \, dV}_{\text{ZDROJE/PROPADY}}$$

P objemový zdroj
$[m^3/m^3/s]$

∂V
$\Downarrow$
OBJEMOVÝ (GAUSS.V.)

$$\int_{\partial V} \rho \vec{q} \, dV = \int_V div(\rho \vec{q}) \, dV$$

$$\nabla \cdot (\rho \vec{q})$$

$$\int_V \frac{d}{dt}(\rho n) \, dV + \int_V \nabla \cdot (\rho \vec{q}) \, dV - \int_V P \rho \, dV = 0$$

Arbitrary dV

$$\int_V \left[\frac{d}{dt}(\rho n) - \qquad \right] dV = 0$$

LIBOVOLNÝ V
$\Downarrow$
INTEGRAND = 0

$$\boxed{\frac{d}{dt}(\rho n) + \nabla \cdot (\rho \vec{q}) = P \rho}$$

STLAČITELNOST compressibility

Both fluid and solid matric are incompressible

A) NESTLAČITELNÁ TEKUTINA I MATRICE → $\frac{d}{dt}(\rho n) = 0$

ROVNICE $\nabla \cdot \vec{q} = P$

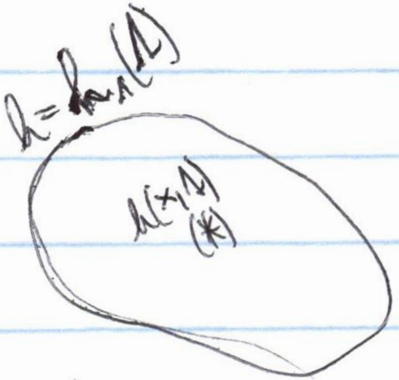
DOSAZENÍ ... $\nabla \cdot (-K \nabla h) = P$

$-\nabla (K \nabla h)$

K KONST. ... $-K \cdot \Delta h = P$ (*) POTENCIÁLOVÉ POLE

Potential field

$h(\vec{x})$



PDR ELIPTICKÁ

Elliptic partial differential equation (PDE)

Instant reaction to external condition change = “perfect inelastic”

IDEALIZACE.... OKAMŽITÁ REAKCE NA ZMĚNU VNĚJŠÍCH PODM.

... „IDÁLNĚ NEPRUŽNÝ“

APROX: SEKVENCE STACIONÁRNÍCH STAVŮ

Model approximation as a sequence of steady states

Compressible case

B) STLAČITELNÉ
TEKUT. ρ — Changing fluid density
MATRICE n — Changing porosity

$$n\frac{d\rho}{dt}+\rho\frac{dn}{dt}+\nabla\cdot(\rho\vec{q})=P\rho$$

STLAČ. $\rho(p)$, $n(p)$ — Constitutive relations — KONSTITUČNÍ VZTAHY

$$\frac{d\rho}{dt}=\frac{d\rho}{dp}\cdot\frac{dp}{dt} \qquad \frac{dp}{dt} \sim \frac{dh}{dt}$$

$$\alpha\cdot\frac{dh}{dt}+\beta\frac{dh}{dt}$$

$$\boxed{\rho S_0\frac{dh}{dt}+\nabla(\rho\vec{q})=P\rho}$$

Specific storativity

S_0 ... SPECIFICKÁ STORATIVITA

VODA ... POMINUJE STLAČITELNOST MATRICE (POHYB ZRN VŮČI SOBĚ)

Matrix compressibility dominates for water (movement of grains)

$$\boxed{S_0\frac{dh}{dt}-\nabla\cdot(K\nabla h)=P}$$

PDR PARABOLICKÁ

Parabolic PDE

Boundary conditions

PDR 2. ŘÁDU OBECNÉ

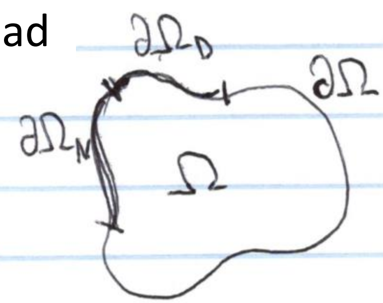
Generic 2nd order PDE

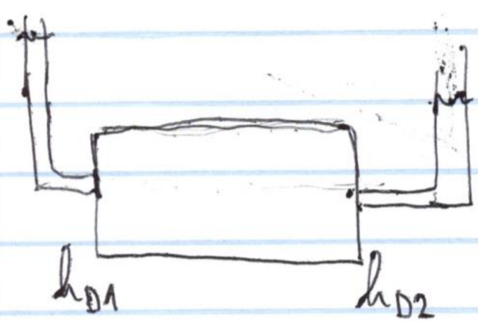
- DIRICHLET (1. DR.) NEZN. Primary unknown h, p
- NEUMANN (2. DR.) DERIV. derivative $\nabla h \ldots \vec{q}$
- 3. DR. NEZN. + DERIV.

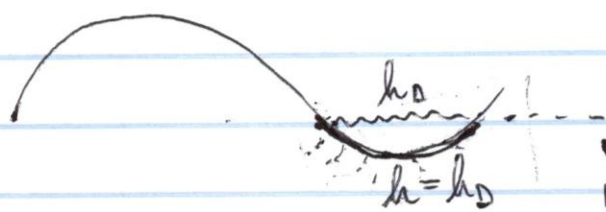
DIRICHLET Prescribed pressure or piezometric head

= PŘEDEP. TLAK / HYDR. VÝŠKA

$$h(\vec{x}, t) = h_D(\vec{x}, t) \quad \vec{x} \in \partial\Omega_D$$

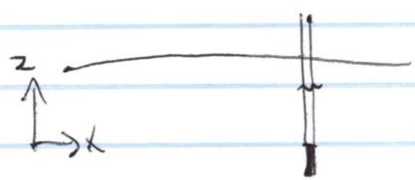






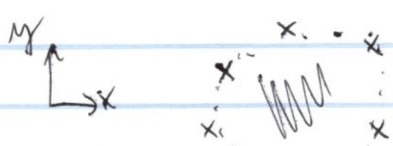
VÝŠKA HLADINY POVRCHU VODY

Water level



MĚŘENÍ VE VRTU

Borehole measurement



EXTRAPOLACE Z ŘADY VRTŮ

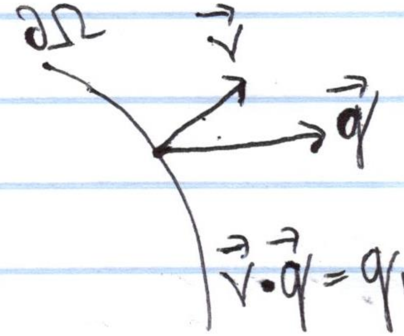
Extrapolation of borehole points

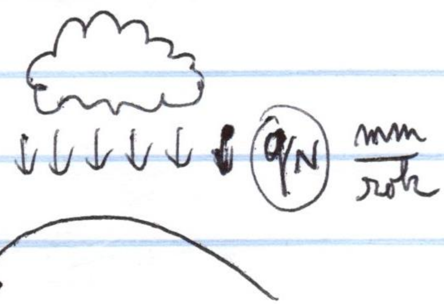
NEUMANN

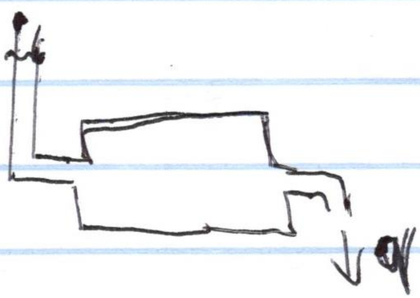
DEFINACE ⇒ TOK

Prescribed flux

$$-\vec{\nabla}\cdot(K\nabla h) = q_N$$



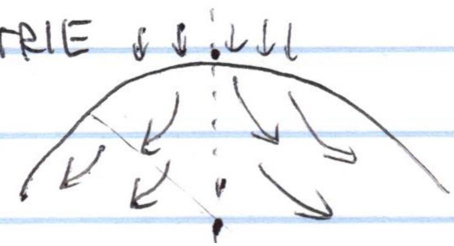




NULOVÝ TOK $-\vec{\nabla}\cdot(K\nabla h) = 0$

- IZOLOVANÁ HRANICE

- SYMETRIE



Zero flux for

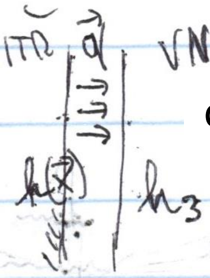
- Isolated boundary
- symmetry

3rd kind b.c.

O.P. 3. DRUHU (CAUCHY, ~~NEUM.~~ NEWTON)

UVNITŘ

inside

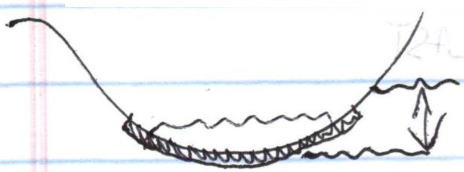


VNĚ

outside

HRANICE = POLOPROPUSTNÁ VRSTVA

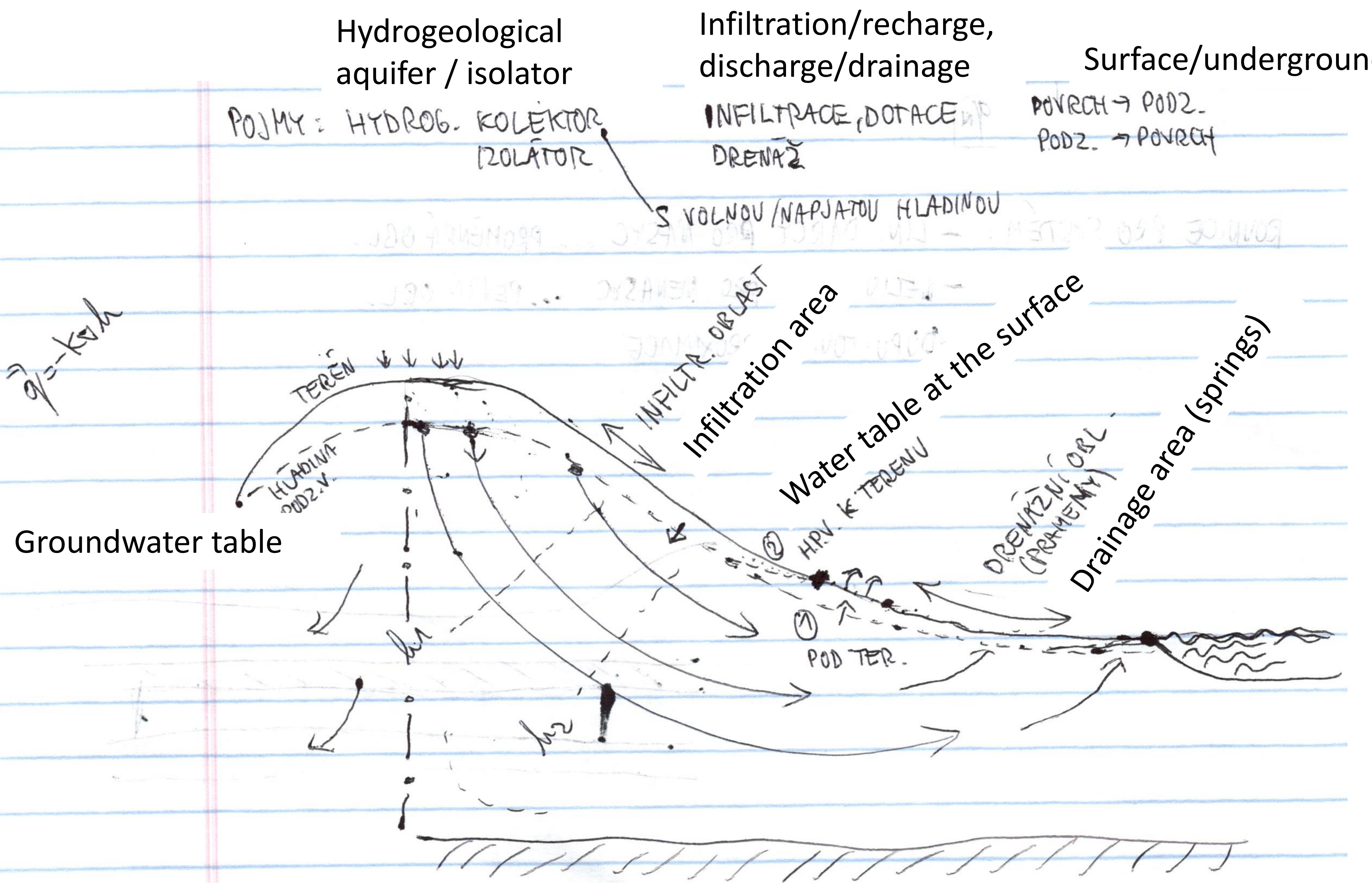
Boundary = semipermeable layer

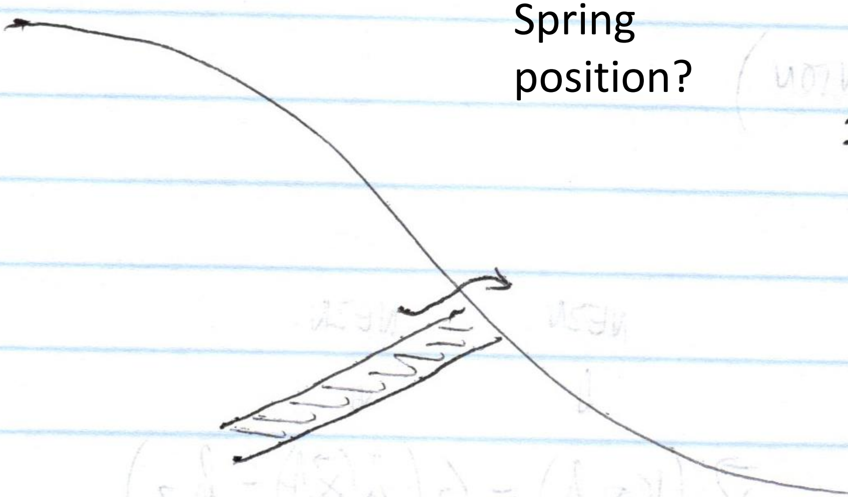


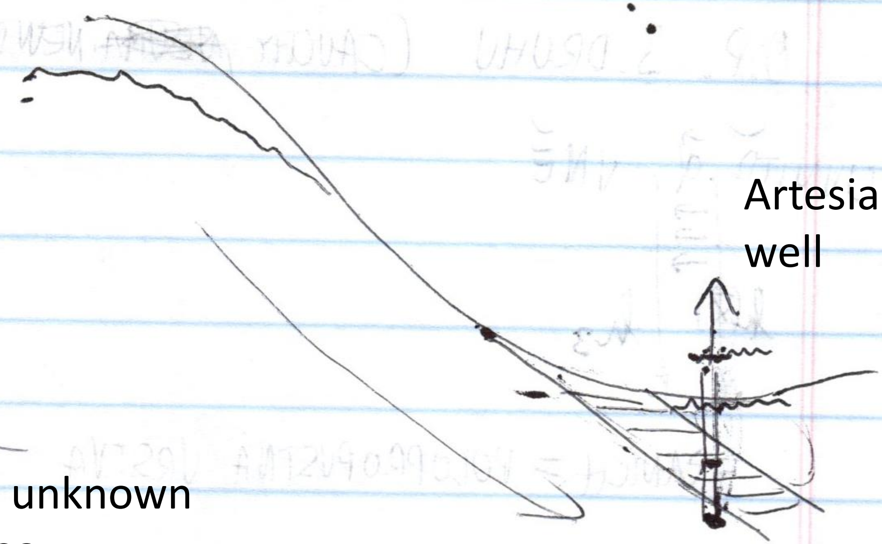
$$-\vec{\nabla}\cdot(\underbrace{K}_{\text{KOEF.}}\nabla \overbrace{h}^{\text{NEZN.}}) = \underbrace{\sigma}_{\text{KOEF.}}(\overbrace{h(\vec{x},t)}^{\text{NEZN.}} - \underbrace{h_3}_{\text{ZADANÁ}})$$

given

Examples – natural groundwater systems







Boundary condition for unknown infiltration/drainage area

OKRAJOVÁ PODMÍNKA – NEZNÁMÁ INFILT. / DREN. OBLAST

„SEEPAGE FACE"

NELIN. – ZÁVISÍ NA HODNOTĚ ŘEŠENÍ $h(\vec{x})$ $\vec{q}(\vec{x})$

Nonlinear – depends on unknown quantities

$\boxed{= q_N < 0}$

INFILTR.: DÁNO $\vec{q}\cdot\vec{\nu} < 0$ $p < 0$ NEZN.

DREN. NEZN. $\vec{q}\cdot\vec{\nu} > 0$ $p = 0$ DÁNO

$\boxed{q_N}$

variants:
Saturated … linear… variable domain
Unsaturated … nonlinear … fixed dom.
Dupuit approximation

ROVNICE PRO SYSTÉM: – LIN. DARCY PRO NASYC. ... PROMĚNNÁ OBL.
– NELIN. PRO NENASYC. ... PEVNÁ OBL.
– DUPUITOVA APROXIMACE