Nevlastní integrál.

a) vlivem meze

$$\int_a^{+\infty} f(x)\,dx$$

b) vlivem fce

$$\int_a^b f(x)\,dx \quad , \text{kde } \lim_{x \to b-} |f(x)| = +\infty$$

1c / cv13

$$\int_0^{+\infty} \frac{2x}{x^2+1}\,dx = \left| \begin{array}{l} \boxed{t = x^2+1} \\ \frac{dt}{dx} = 2x \\ dx = \frac{dt}{2x} \end{array} \right| = \int_1^{+\infty} \frac{\cancel{2x}}{t} \cdot \frac{dt}{\cancel{2x}}$$

$x = 0 \rightarrow t = 1$

$x = +\infty \rightarrow t = +\infty$

$$= \int_1^{+\infty} \frac{1}{t}\,dt = \lim_{s \to +\infty} \int_1^{s} \frac{1}{t}\,dt = \lim_{s \to +\infty} \Big[\ln|t| \Big]_{t=1}^{s}$$

$$= \lim_{s \to +\infty} \ln s - \ln|1| = \lim_{s \to +\infty} \ln s = \underline{\underline{+\infty}}$$ ... diverguje

Rychlejší výpočet

$$\int_0^{+\infty} \frac{2x}{x^2+1}\,dx = \Big[\ln|x^2+1|\Big]_{x=0}^{+\infty} = \ln|(+\infty)^2+1| - \ln|1|$$

$$= \ln(\infty) - \ln(1) = +\infty - 0 = \underline{\underline{+\infty}}$$

2a

$$\int_0^2 \frac{2}{\sqrt{4-x^2}}\,dx = \lim_{r \to 2-} \int_0^{r} \frac{2}{\sqrt{4-x^2}}\,dx = (*)$$

$$\int \frac{2}{\sqrt{4-x^2}}\,dx = \frac{1}{\cancel{2}} \int \frac{\cancel{2}}{\sqrt{1-\frac{x^2}{4}}}\,dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

$$\left| \begin{array}{l} t = \frac{x}{2} \\ dt = \frac{1}{2}\,dx \\ dx = 2\,dt \end{array} \right| = \int \frac{1}{\sqrt{1-t^2}}\,2\,dt = 2 \cdot \arcsin(t) = 2 \cdot \arcsin\left(\frac{x}{2}\right) + C$$

$$(*) = \lim_{t \to 2-} \left[2 \cdot \arcsin\left(\frac{x}{2}\right) \right]_{x=0}^{t} = \lim_{t \to 2-} 2 \cdot \arcsin\left(\frac{t}{2}\right) - \underbrace{2 \cdot \arcsin\left(\frac{0}{2}\right)}_{0}$$

$$= 2 \cdot \arcsin(1-) = 2 \cdot \frac{\pi}{2} = \pi$$ ... konverguje

Rychlejší postup:

$$\int_0^2 \frac{2}{\sqrt{4-x^2}}\, dx = \left[2 \cdot \arcsin\left(\frac{x}{2}\right) \right]_{x=0}^{2} = 2 \arcsin(1) - 2 \arcsin 0$$

$$= 2 \cdot \frac{\pi}{2} = \pi$$

3a $\displaystyle\int_1^{+\infty} \frac{1}{x^2-1}\,dx = \underbrace{\int_1^2 \frac{1}{x^2-1}\,dx}_{I_1} + \underbrace{\int_2^{+\infty} \frac{1}{x^2-1}\,dx}_{I_2}$

$$\int \frac{1}{\underbrace{x^2-1}_{(x+1)(x-1)}}\,dx = \int \frac{-1/2}{x+1} + \frac{1/2}{x-1}\,dx = -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1|$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

$$I_1 = \lim_{t \to 1+} \int_t^2 \frac{1}{x^2-1}\,dx = \lim_{t \to 1+} \left[\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_{x=t}^{2}$$

$$= \frac{1}{2}\ln\left(\frac{1}{3}\right) - \lim_{t \to 1+} \frac{1}{2}\ln\left|\underbrace{\frac{t-1}{t+1}}_{\to 0^+}\right| = \frac{1}{2}\ln\left(\frac{1}{3}\right) - \frac{1}{2}\ln(0+)$$

$$= \frac{1}{2}\ln\left(\frac{1}{3}\right) - (-\infty) = \underline{\underline{\frac{1}{2}\ln\left(\frac{1}{3}\right) + \infty}}$$

$$I_2 = \lim_{s \to +\infty} \int_2^s \frac{1}{x^2-1}\,dx = \lim_{s \to +\infty} \left[\frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| \right]_{x=2}^{s}$$

$$= \lim_{s \to +\infty} \frac{1}{2} \ln \left| \frac{s-1}{s+1} \right| - \frac{1}{2} \ln \left| \frac{1}{3} \right| = \underbrace{\frac{1}{2} \ln 1}_{=0} - \frac{1}{2} \ln \left| \frac{1}{3} \right|$$

$$\lim_{s \to +\infty} \frac{s-1}{s+1} = 1$$

$$= - \frac{1}{2} \ln \left| \frac{1}{3} \right|$$

$$I_1 + I_2 = \cancel{\frac{1}{2} \cdot \ln \left| \frac{1}{3} \right|} + \infty - \cancel{\frac{1}{2} \ln \left| \frac{1}{3} \right|} = +\infty \text{ ...divergira}$$

Rychlejší způsob:

$$\int_1^{+\infty} \frac{1}{x^2-1}\,dx = \left[\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_{x=1}^{+\infty}$$

$$= \frac{1}{2}\ln(1) - \frac{1}{2}\ln(0+) = 0 - (-\infty) = \underline{\underline{+\infty}}$$

$$\lim_{x\to+\infty} \frac{x-1}{x+1}$$

diverguje