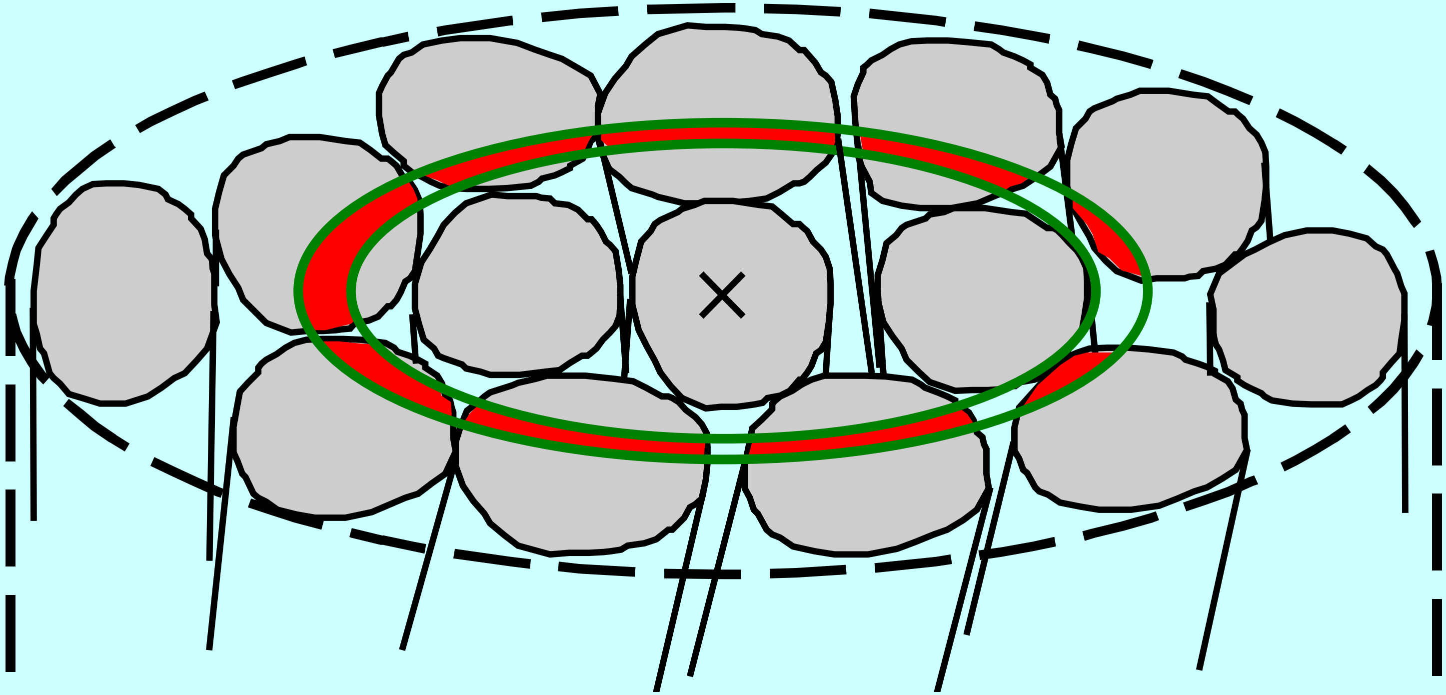
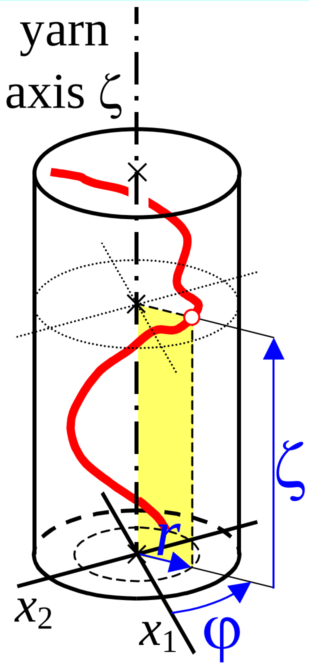
MODELING OF INTERNAL YARN GEOMETRY



The practically useful yarn properties are the result of fiber properties, the mutual fiber interactions inside the yarn, and the interactions between yarn and outer influences. The internal structure of yarn is very important especially for geometrical and mechanical properties of yarn.
We observe that the specific regulations of internal yarn geometry are relatively complicated due to the complex nature of deterministic and random influences. Lots of mathematical models on this topic were created during the last two centuries. Some of the well-known model will be presented in this lecture.

GENERAL DESCRIPTION OF FIBERS IN YARN

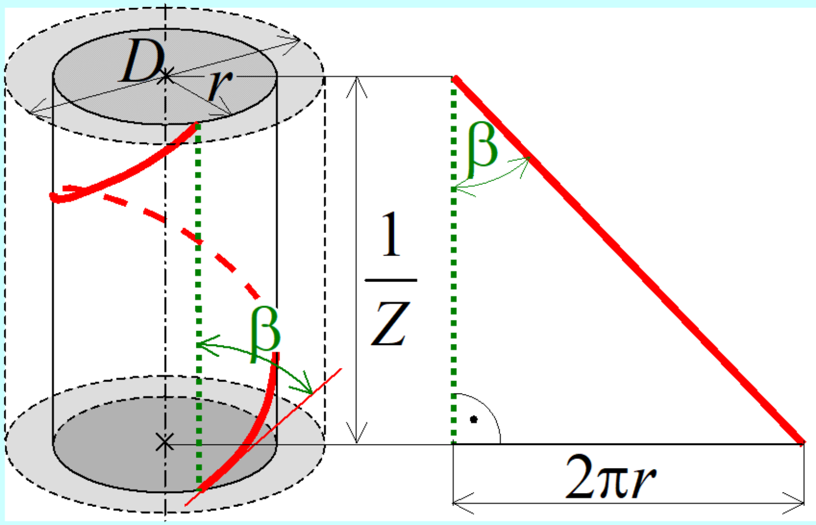
General fiber trajectories in yarn:
- **complicated shapes (/)**
- **random characters**
- **deterministic trends**



Description of point on fiber : **cylindrical coordinates** r, φ, ζ

HELICAL MODEL OF YARN

Assumptions of helical model can be formulated in the following way:

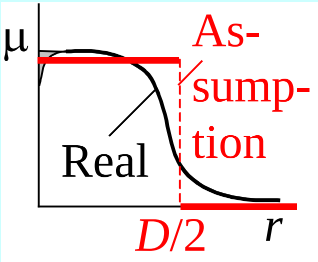


1. **Helical paths of fibers** (same sense of rotation).

2. **Common axis** of all helixes is yarn axis.

3. **Same coil height** for all fibers. Height of one fiber coil is $1/Z$, **same** for all fibers.

The fiber positions in yarn (“starting points”), which are not determined by the assumptions stated before, can be characterized by the radial function of packing density $\mu = \mu(r)$. Because it is relatively complicated, the following assumption are used:



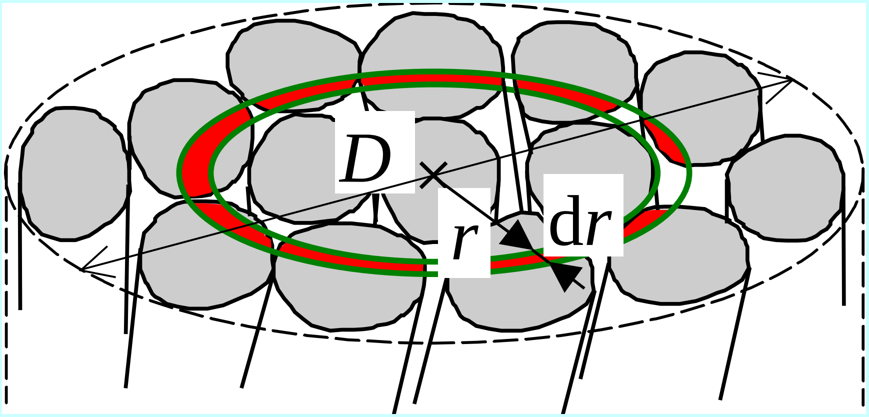
4. **Packing density is constant** in all places inside the yarn.

Note: If all these 4 assumptions are valid, then we speak about **ideal** helical model.

Note: Helical models and their applications are the oldest concepts of yarn modeling, adherent with the names *A.Köchlin (1828), E.Müller (1880), S.Marschik (1904), A.CH. Gegauff (1907), R. Schwarz (1933), E. Braschler (1935), V.I. Budnikov (1945), L.R.G.Treloar (1956),* etc.

Number of fibers and shortening of yarn

Let us create differential annulus - "<u>differential layer</u>" (*Braschler*) - at the general radius r in a cross-section of **general helical model** of yarn.



<u>Area of differential annulus</u>: $2\pi r\,\mathrm{d}r$

<u>Packing density of differential annulus</u>: $\mu = \mathrm{d}S/(2\pi r\,\mathrm{d}r)$

where $\mathrm{d}S$...area of fiber sections in diff. annulus (red)

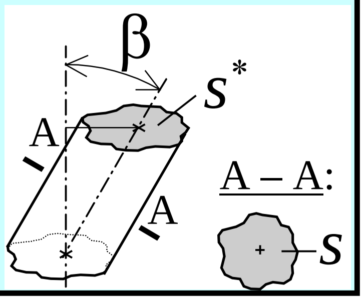
(See lecture 1 - areal interpretation of μ)

<u>Area of fiber sections in differential annulus</u>:

$$\mathrm{d}S = 2\pi\mu r\,\mathrm{d}r$$

Area of oblique section of one fiber: $s^* = s/\cos\beta$

(where s...fiber cross-section - see lecture 1)



Number of fibers in differential annulus:

$$\mathrm{d}n = \overbrace{\mathrm{d}S}^{2\pi r\,\mathrm{d}r\,\mu} \Big/ \overbrace{s^*}^{s/\cos\beta} = \frac{2\pi r\,\mathrm{d}r\,\mu}{s/\cos\beta}, \qquad \mathrm{d}n = 2\pi\cos\beta\,\mu r\,\mathrm{d}r/s,$$

Substantial cross-sectional area of yarn:

$$S = \int_{r=0}^{r=D/2} \overbrace{\mathrm{d}S}^{=2\pi r\,\mathrm{d}r\,\mu} = 2\pi \int_{r=0}^{r=D/2} \mu r\,\mathrm{d}r$$

Mean packing density of yarn:

$$\bar{\mu} = \frac{S}{\pi D^2/4} = \frac{2\pi\int_0^{D/2}\mu r\,\mathrm{d}r}{\pi D^2/4} = \frac{8}{D^2}\int_0^{D/2}\mu r\,\mathrm{d}r$$

Number of fibers in yarn cross-section:

$$n = \int_{r=0}^{r=D/2}\mathrm{d}n = \frac{2\pi}{s}\int_0^{D/2}\overbrace{\cos\beta}^{=\sqrt{1/(1+\tan^2\beta)}}\,\mu r\,\mathrm{d}r = \frac{2\pi}{s}\int_0^{D/2}\frac{\mu r\,\mathrm{d}r}{\sqrt{1+\underbrace{\tan^2\beta}_{(2\pi r Z)^2}}}, \qquad n = \frac{2\pi}{s}\int_0^{D/2}\frac{\mu r\,\mathrm{d}r}{\sqrt{1+(2\pi r Z)^2}}$$

It was also derived before (lecture 1): $n = \tau k_n$, $\tau = T/t = S/s$
(T...yarn count, t...fiber fineness, τ...relative yarn fineness)

Coefficient k_n is now

$$k_n = \frac{n}{\tau} = \frac{s}{S} n = \frac{s}{S} \overbrace{\frac{2\pi}{s} \int_0^{D/2} \frac{\mu r \, \mathrm{d}r}{\sqrt{1+(2\pi r Z)^2}}}^{=n}, \qquad k_n = \frac{2\pi}{S} \int_0^{D/2} \frac{\mu r \, \mathrm{d}r}{\sqrt{1+(2\pi r Z)^2}}$$

Note: The relation $\mu = \mu(r)$ is necessary to know for numerical calculation of $S, \bar{\mu}, n, k_n$. It is possible to obtain the function $\mu(r)$ as a result of experiment or try to apply some theoretical model (e.g. based on differential equation of radial forces equilibrium in yarn – *V.I.Budnikov, J.W.S.Hearle, B.Neckář* etc.)

Ideal helical model satisfies the assumption $\mu = \text{const.}$ and then substantial cross-sectional area of yarn:

$$S = 2\pi \int_{r=0}^{r=D/2} \overbrace{\mu}^{=\text{const.}} r \, \mathrm{d}r = 2\pi\mu \left(\frac{D^2/4}{2} \right), \qquad S = \left(\pi D^2/4 \right) \mu$$

Mean packing density of yarn:

$$\bar{\mu} = \frac{8}{D^2}\int_0^{D/2} \overbrace{\mu}^{=\text{const.}}\, r\,\mathrm{d}r = \frac{8\mu}{D^2}\left(\frac{D^2/4}{2}\right), \quad \bar{\mu} = \mu$$

Further, the following integral is valid:

$$I = \int_0^{D/2} \frac{r\,\mathrm{d}r}{\sqrt{1+(2\pi r Z)^2}} = \int_1^{\sqrt{1+(\pi D Z)^2}} \frac{x\,\mathrm{d}x}{x(2\pi Z)^2} = \frac{1}{(2\pi Z)^2}\left[\sqrt{1+(\pi D Z)^2} - 1\right]$$

Substitution: $x^2 = 1 + (2\pi r Z)^2$,

$$2x\,\mathrm{d}x = (2\pi Z)^2\, 2r\,\mathrm{d}r, \quad r\,\mathrm{d}r = x\,\mathrm{d}x / (2\pi Z)^2$$

Number of fibers in yarn cross-section:

$$n = \frac{2\pi}{s}\int_0^{D/2} \frac{\overbrace{\mu}^{=\text{const.}}\, r\,\mathrm{d}r}{\sqrt{1+(2\pi r Z)^2}} = \frac{2\pi\mu}{s}\overbrace{\int_0^{D/2} \frac{r\,\mathrm{d}r}{\sqrt{1+(2\pi r Z)^2}}}^{=I} = \frac{2\pi\mu}{s}\frac{1}{(2\pi Z)^2}\left[\sqrt{1+(\pi D Z)^2} - 1\right] =$$

$$= \frac{2}{(\pi D Z)^2}\overbrace{\left\{\overbrace{(\pi D^2/4)}^{=S}\mu / s\right\}}^{=\tau}\left[\sqrt{1+(\pi D Z)^2} - 1\right], \quad n = \frac{2\tau}{(\pi D Z)^2}\left[\sqrt{1+(\pi D Z)^2} - 1\right]$$

Coefficient k_n is now

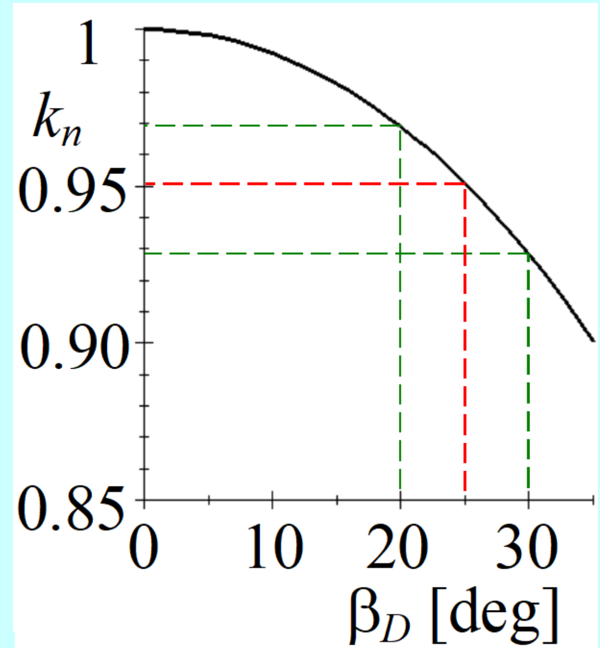
$$k_n = \frac{n}{\tau} = \frac{1}{\tau}\overbrace{\frac{2\tau}{(\pi DZ)^2}\left[\sqrt{1+(\pi DZ)^2}-1\right]}^{=n}, \qquad k_n = \frac{2}{(\pi DZ)^2}\left[\sqrt{1+(\pi DZ)^2}-1\right]$$

For the fiber on the yarn surface ($r = r_D,\ \beta = \beta_D$) it is valid

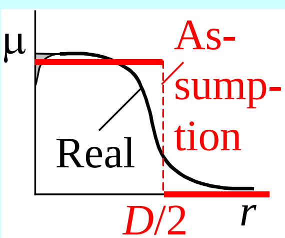
$\tan\beta_D = 2\pi \overbrace{r}^{=D/2} Z = \pi DZ = \kappa$ ...intensity of twist (see also the derivation in lecture 1). The alternative equation for k_n can be derived using equation mentioned before.

$$k_n = \frac{2}{(\underbrace{\pi DZ}_{=\tan\beta_D})^2}\left[\sqrt{1+(\underbrace{\pi DZ}_{=\tan\beta_D})^2}-1\right] = \frac{2}{\left(\underbrace{\tan\beta_D}_{=\sin\beta_D/\cos\beta_D}\right)^2}\left[\sqrt{\underbrace{1+\tan^2\beta_D}_{=1/\cos^2\beta_D}} - \underbrace{1}_{=\cos\beta_D/\cos\beta_D}\right] =$$

$$= 2\frac{\cos^2\beta_D}{\sin^2\beta_D}\frac{\overbrace{(1-\cos\beta_D)(1+\cos\beta_D)}^{=(1-\cos^2\beta_D)=\sin^2\beta_D}}{\cos\beta_D(1+\cos\beta_D)}, \qquad k_n = \frac{2\cos\beta_D}{1+\cos\beta_D}$$

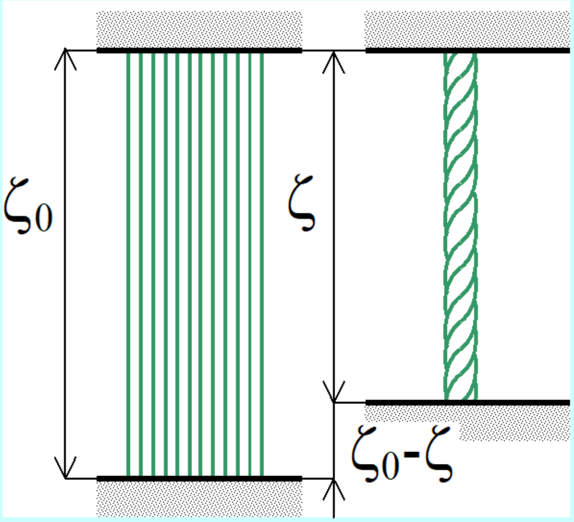


The graphical interpretation of last function do shown on the picture. *Note:* The angles β_D of common yarns lie usually between 20° and 30°. Then the value of k_n surrounding the value 0.95 characterizes the influence of fiber slope due to twist in ideal helical model. The value k_n can be a little higher for real function of $\mu(r)$ (follows our equation for general helical model) and less in consequence of different effects of radial migration. (In reality, we measured the values round 0.95 for traditional ring yarns, but values round 0.80 for rotor yarns.)



YARN RETRACTION IN IDEAL HELICAL MODEL

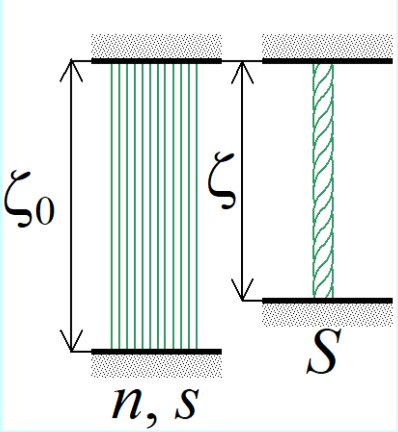
	Non twisted	*Twisted*
Length of bundle	ζ_0	ζ
Yarn retraction	$\delta = (\zeta_0 - \zeta)/\zeta_0 = 1 - \zeta/\zeta_0$	
Number of fibers	n	n
Volume of fibers	V_0	V
Mass of fibers	m	m
Starting yarn count	$T_0 = m/\zeta_0$	
Yarn count (final)	...	$T = m/\zeta$, $T = T_0/(1-\delta)$
Number of coils	0	N_C
Latent yarn twist	$Z_0 = N_C/\zeta_0$	
Yarn twist (real)	...	$Z = N_C/\zeta$, $Z = Z_0/(1-\delta)$
Latent twist coeff.	$\alpha_0 = Z_0\sqrt{T_0}$	
Twist coeff. (real)	...	$\alpha = Z\sqrt{T}$, $\alpha = \alpha_0/(1-\delta)^{3/2}$



2. **Idea of total fiber volume**

Usually the following *assumption* is taken as granted *(E. Braschler)* : Total fiber volume in yarn and fiber cross-sectional area (fiber diameter) do not change due to twist.

$$V_0 = V \text{...const.,} \quad s\text{...const.}$$



Fiber volume - non-twisted $V_0 = n\, s\zeta_0$

- twisted $V = S\zeta$ (lecture 1)

$$\overbrace{V_0}^{=n\,s\zeta_0} = \overbrace{V}^{=S\zeta}, \quad n\,s\zeta_0 = S\zeta, \quad \zeta/\zeta_0 = \overbrace{s\Big/\Big(\overbrace{S/n}^{=\bar{s}^*}\Big)}^{=k_n}, \qquad \zeta/\zeta_0 = k_n$$

Retraction was defined by equation $\delta = 1 - \zeta/\zeta_0$, coefficient k_n was derived before for ideal helical model. Therefore

yarn retraction $$\delta = 1 - \overbrace{\left(\frac{\zeta}{\zeta_0}\right)}^{=k_n} = 1 - \overbrace{k_n}^{=\frac{2}{(\pi DZ)^2}\left[\sqrt{1+(\pi DZ)^2}-1\right]} = 1 - \frac{2}{(\pi DZ)^2}\left[\sqrt{1+(\pi DZ)^2}-1\right]$$

(Continuation)

$$\delta=\frac{\left(\sqrt{1+(\pi DZ)^2}+1\right)-\frac{2}{(\pi DZ)^2}\overbrace{\left[\sqrt{1+(\pi DZ)^2}-1\right]\left(\sqrt{1+(\pi DZ)^2}+1\right)}^{=\left[\sqrt{1+(\pi DZ)^2}\right]^2-1^2}}{\sqrt{1+(\pi DZ)^2}+1}=$$

$$=\frac{\overbrace{\sqrt{1+(\pi DZ)^2}+1-\overbrace{\frac{2}{(\pi DZ)^2}\overbrace{\left\{1+(\pi DZ)^2-1\right\}}^{=(\pi DZ)^2}}^{=2}}^{=-1}}{\sqrt{1+(\pi DZ)^2}+1}$$

$$\delta=\frac{\sqrt{1+(\pi DZ)^2}-1}{\sqrt{1+(\pi DZ)^2}+1}=$$

$$=\frac{\sqrt{1+\tan^2\beta_D}-1}{\sqrt{1+\tan^2\beta_D}+1}$$

In another form

$$\delta=1-\overbrace{\left(\frac{\zeta}{\zeta_0}\right)}^{=k_n}=1-\overbrace{k_n}^{=\frac{2\cos\beta_D}{1+\cos\beta_D}}=\frac{1+\cos\beta_D}{1+\cos\beta_D}-\frac{2\cos\beta_D}{1+\cos\beta_D}=\frac{\overbrace{1+\cos\beta_D-2\cos\beta_D}^{=1-\cos\beta_D}}{1+\cos\beta_D},$$

$$\delta=\frac{1-\cos\beta_D}{1+\cos\beta_D}$$

Rearrangement

$$\delta=\frac{1-\cos\beta_D}{1+\cos\beta_D}=\frac{\underbrace{\left(\cos^2\frac{\beta_D}{2}+\sin^2\frac{\beta_D}{2}\right)}_{=1}-\underbrace{\left(\cos^2\frac{\beta_D}{2}-\sin^2\frac{\beta_D}{2}\right)}_{=\cos\beta_D=\cos\left(2\frac{\beta_D}{2}\right)}}{\underbrace{\left(\cos^2\frac{\beta_D}{2}+\sin^2\frac{\beta_D}{2}\right)}_{=1}+\underbrace{\left(\cos^2\frac{\beta_D}{2}-\sin^2\frac{\beta_D}{2}\right)}_{=\cos\beta_D}}=\frac{2\sin^2\frac{\beta_D}{2}}{2\cos^2\frac{\beta_D}{2}},\qquad \delta=\tan^2\frac{\beta_D}{2}$$

Because $\kappa=\tan\beta_D=\pi DZ=2\sqrt{\pi}\,\alpha\big/\sqrt{\mu\rho}$ (see before),

$$\delta=\frac{\sqrt{1+\left(2\sqrt{\pi}\,\alpha\big/\sqrt{\mu\rho}\right)^2}-1}{\sqrt{1+\left(2\sqrt{\pi}\,\alpha\big/\sqrt{\mu\rho}\right)^2}+1},\qquad \delta=\frac{\sqrt{1+4\pi\alpha^2/(\mu\rho)}-1}{\sqrt{1+4\pi\alpha^2/(\mu\rho)}+1}$$

Using $\alpha=\alpha_0\big/(1-\delta)^{3/2}$, we can express δ as a function of latent twist coefficient α_0 as follows

$$\delta=\left[\sqrt{1+4\pi\left(\alpha_0\big/(1-\delta)^{3/2}\right)^2\big/(\mu\rho)}-1\right]\Big/\left[\sqrt{1+4\pi\left(\alpha_0\big/(1-\delta)^{3/2}\right)^2\big/(\mu\rho)}+1\right],$$

(Continuation)

$$\delta = \left[\sqrt{1+\frac{4\pi\alpha_0^2}{\mu\rho(1-\delta)^3}}-1\right]\Big/\left[\sqrt{1+\frac{4\pi\alpha_0^2}{\mu\rho(1-\delta)^3}}+1\right]$$

If we consider $\frac{4\pi\alpha_0^2}{\mu\rho(1-\delta)^3} = A$, then we can write

$$\delta = \left[\sqrt{1+A}-1\right]\Big/\left[\sqrt{1+A}+1\right],\quad \delta\sqrt{1+A}+\delta = \sqrt{1+A}-1,\quad 1+\delta = -\delta\sqrt{1+A}+\sqrt{1+A},$$

$$1+\delta = \sqrt{1+A}(1-\delta),\quad [1+\delta]^2 = \left[\sqrt{1+A}(1-\delta)\right]^2,\quad 1+2\delta+\delta^2 = (1+A)(1-\delta)^2,$$

$$\left[1+2\delta+\delta^2\right](1-\delta) = (1+A)(1-\delta)^2(1-\delta),\quad \left[1+2\delta+\delta^2\right](1-\delta) = (1+A)(1-\delta)^3,$$

$$1+2\delta+\delta^2-\delta-2\delta^2-\delta^3 = (1-\delta)^3\left[1+\underbrace{\left(\frac{4\pi\alpha_0^2}{\mu\rho(1-\delta)^3}\right)}_{=A}\right] = \underbrace{1-3\delta+3\delta^2-\delta^3}_{=(1-\delta)^3}+4\pi\alpha_0^2/(\mu\rho),$$

$$\delta-\delta^2 = -3\delta+3\delta^2+4\pi\alpha_0^2/(\mu\rho),\quad 0 = 4\delta^2-4\delta+4\pi\alpha_0^2/(\mu\rho),\quad 0 = \delta^2-\delta+\pi\alpha_0^2/(\mu\rho),$$

$$\delta = \frac{1\pm\sqrt{1-\frac{4\pi}{\mu}\frac{\alpha_0^2}{\rho}}}{2},$$

$$\delta = \frac{1}{2}\pm\frac{1}{2}\sqrt{1-\frac{4\pi}{\mu}\frac{\alpha_0^2}{\rho}}$$

The discriminant of the quadratic equation must not be negative. Therefore $1-\frac{4\pi}{\mu}\frac{\alpha_0^2}{\rho}\geq 0,\quad \alpha_0^2\leq\frac{\mu\rho}{4\pi},$ $\frac{\alpha_0}{\sqrt{\mu\rho}}\leq\frac{1}{\sqrt{4\pi}}$

The latent twist coefficient is limited!

<u>Limit case</u>: $\frac{\alpha_0}{\sqrt{\mu\rho}}=\frac{1}{\sqrt{4\pi}}=0.281$ $\delta=\frac{1}{2}\pm\frac{1}{2}\sqrt{1-\underbrace{\frac{4\pi}{\mu}\frac{\alpha_0^2}{\rho}}_{=0}},$ $\delta=\frac{1}{2}$

Since $\delta=\tan^2\frac{\beta_D}{2}$, hence $\frac{1}{2}=\tan^2\frac{\beta_D}{2},\ \beta_D=2\arctan\sqrt{\frac{1}{2}},$ $\beta_D=70.5^\circ$

From yarn retraction $\delta=\left(\sqrt{1+\tan^2\beta_D}-1\right)\Big/\left(\sqrt{1+\tan^2\beta_D}+1\right)$

we obtain $\delta\sqrt{1+\tan^2\beta_D}+\delta=\sqrt{1+\tan^2\beta_D}-1,$

$$1+\delta=\sqrt{1+\tan^2\beta_D}\,(1-\delta),\quad \left(\frac{1+\delta}{1-\delta}\right)^2=1+\tan^2\beta_D,\quad \tan\beta_D=\sqrt{\left(\frac{1+\delta}{1-\delta}\right)^2-1}$$

and using the value $\delta = 1/2$ we get for the limit case

$$\overbrace{\tan\beta_D}^{=\kappa=\pi DZ} = \sqrt{\left(\frac{1+0.5}{1-0.5}\right)^2 - 1} = \sqrt{3^2-1}, \quad \kappa = \pi DZ = \tan\beta_D = 2\sqrt{2} = 2.828$$

The value of twist intensity is limited!

Comparison of ideas

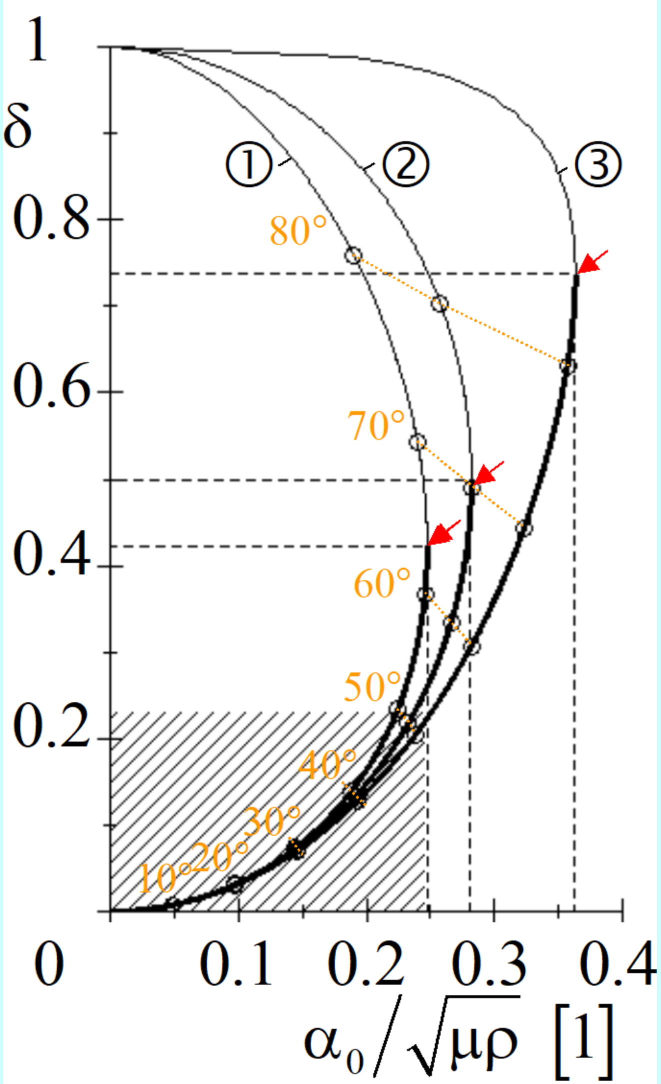
Summary of equations derived before:

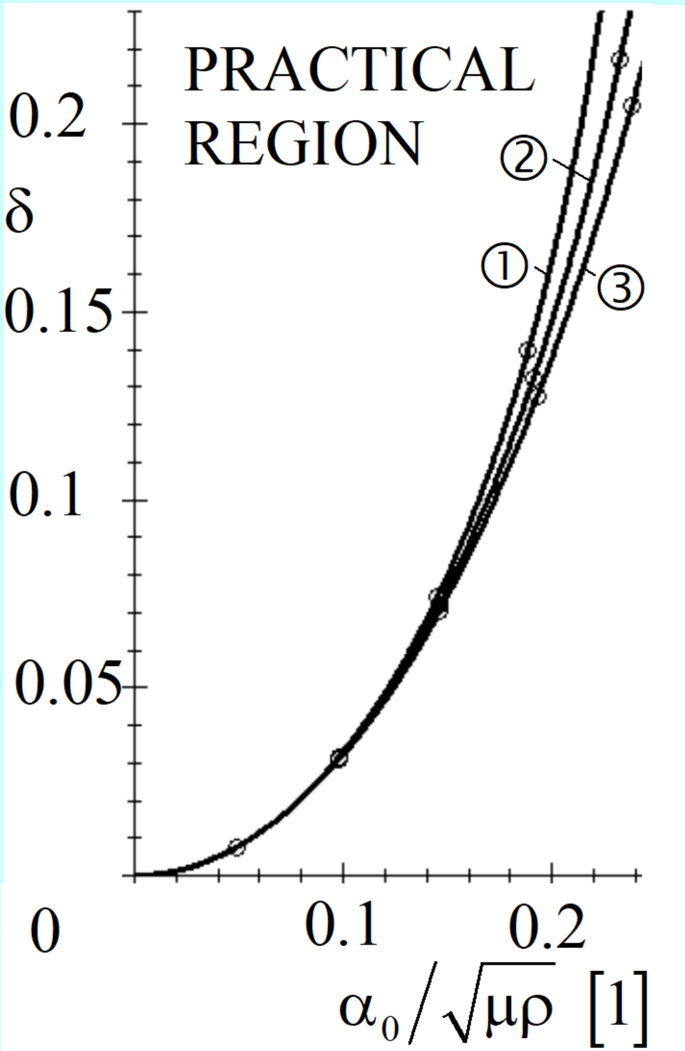
Idea	Retraction
① Neutral radius	$\delta = 1 - \frac{1}{\sqrt{1+2\pi\alpha^2/(\mu\rho)}} = 1 - \frac{2}{\sqrt{3}}\cos\left[\frac{\pi}{3} \pm \frac{1}{3}\arccos\left(\frac{3\sqrt{3}\,\pi}{\mu}\frac{\alpha_0^2}{\rho}\right)\right]$
② Fiber volume	$\delta = \frac{\sqrt{1+4\pi\alpha^2/(\mu\rho)}-1}{\sqrt{1+4\pi\alpha^2/(\mu\rho)}+1} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{4\pi}{\mu}\frac{\alpha_0^2}{\rho}}$
③ Axial force	$\delta = 1 - \frac{\ln\sqrt{1+4\pi\alpha^2/(\mu\rho)}}{\sqrt{1+4\pi\alpha^2/(\mu\rho)}-1} = 1 - \frac{\ln\sqrt{1+\kappa^2}}{\sqrt{1+\kappa^2}-1}, \quad \frac{\alpha_0}{\sqrt{\mu\rho}} = \frac{1}{2\sqrt{\pi}}\left(\frac{\ln\sqrt{1+\kappa^2}}{\sqrt{1+\kappa^2}-1}\right)^{3/2}$ $\kappa = \tan\beta_D$...parameter (twist intensity)

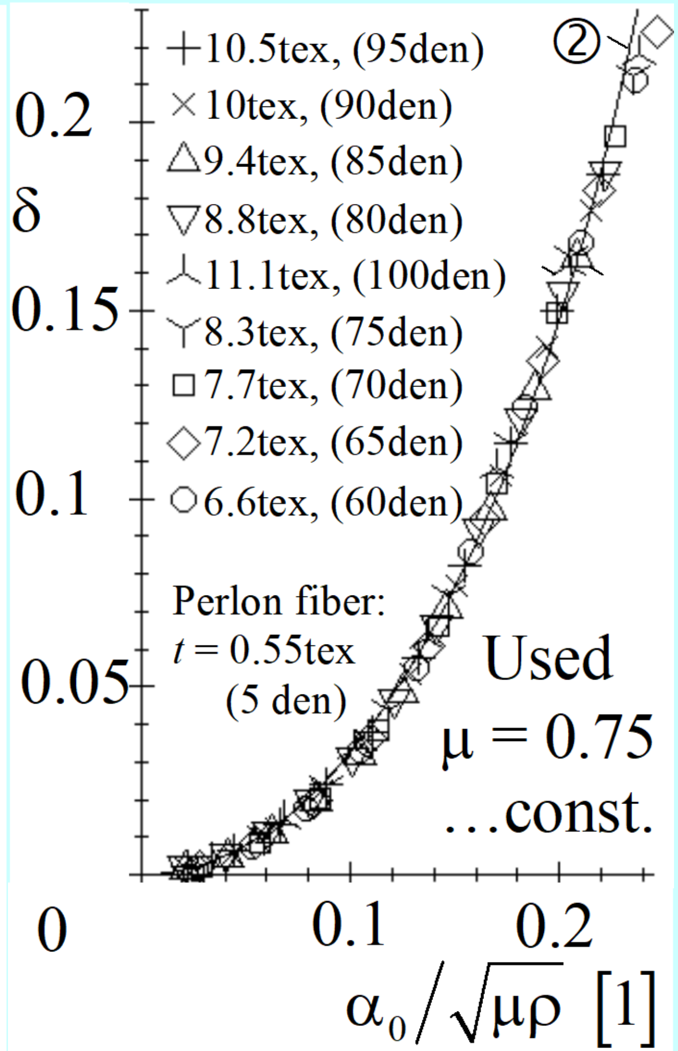
Graphical interpretation

①...Idea of neutral radius
②...Idea of fiber volume
③...Idea of axial force
Thick part ... real
Thin part ... hypothetical
↙ ...limit case - "saturated twist", parameters (summary):

Idea	$\frac{\alpha_0}{\sqrt{\mu\rho}}$	δ	κ	β_D
① Neutral radius	0.248	0.423	2	63°
② Fiber volume	0.281	0.5	2.828	70.5°
③ Axial force	0.363	0.737	9.528	84°







Experiment
(J. Marko, B. Neckář)

Perlon filament yarn

Note:
It is used

$$\alpha_{0\left[kg^{1/2}\,m^{-3/2}\right]} = Z_{0\left[m^{-1}\right]} \cdot \sqrt{T_{0[tex]}}/1000,$$

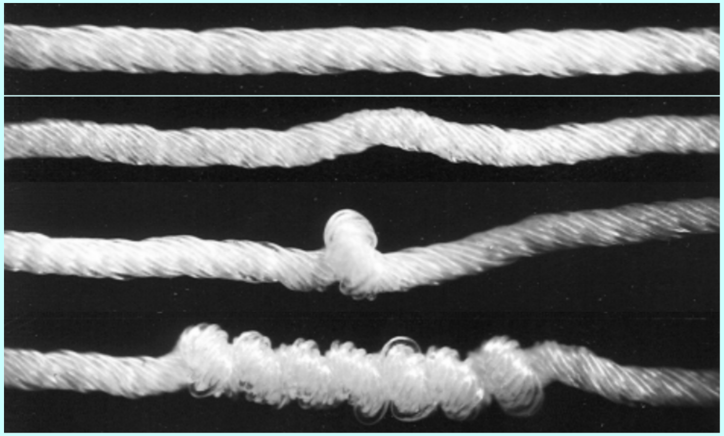
$\rho\ldots\left[kg\,m^{-3}\right]$

Twist of second order

The **saturated twist** (limit case) is really observed experimentally, but for a smaller value of latent twist (usually by β_D from 45° to 55° in case of filament yarns). Axial asymmetry of twisted yarn is the probable reason of this phenomenon.

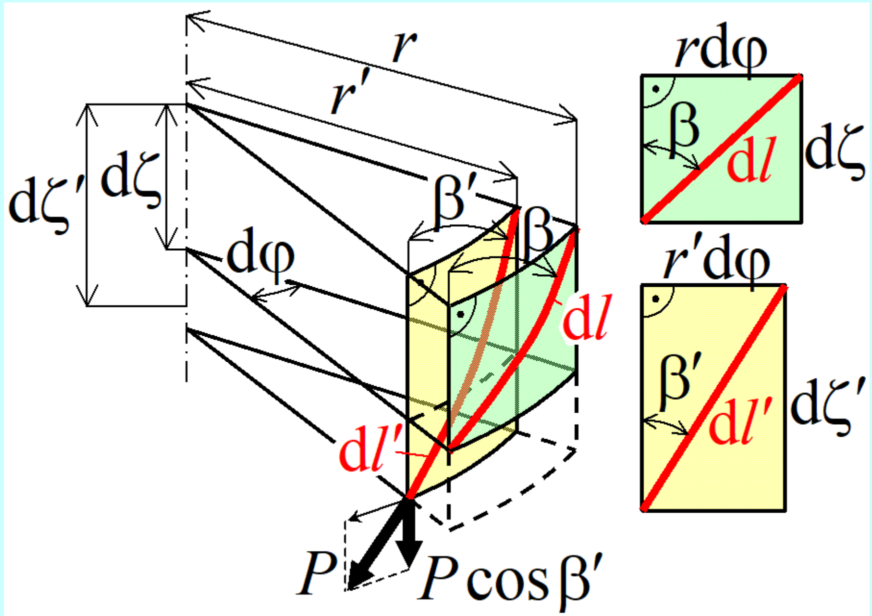
But, what happens if we give higher than the saturated twist to the yarn? The yarn is not able to absorb it "inside" its structure and then some coils will be placed "outside", as coils of **twist of second order**.

The phases of this process are shown here.



Yarn stress-strain relation

like *Gegauff* and others



The general element of the helical fiber (length $\mathrm{d}l$, angle β) lying at the radius r determines an elementary cylindrical surface (green) with dimensions $r\,\mathrm{d}\varphi,\ \mathrm{d}\zeta$. After yarn elongation the same element shifts itself to a new (yellow) position at a smaller radius r' with new angle β' and new dimensions $r'\,\mathrm{d}\varphi,\ \mathrm{d}\zeta'$. It is valid: $\tan\beta = r\,\mathrm{d}\varphi/\mathrm{d}\zeta,\ \tan\beta' = r'\,\mathrm{d}\varphi/\mathrm{d}\zeta'$

<u>Yarn axial strain</u>: $\varepsilon_a = \left(\mathrm{d}\zeta' - \mathrm{d}\zeta\right)/\mathrm{d}\zeta = \mathrm{d}\zeta'/\mathrm{d}\zeta - 1,\quad \mathrm{d}\zeta' = \left(1+\varepsilon_a\right)\mathrm{d}\zeta$

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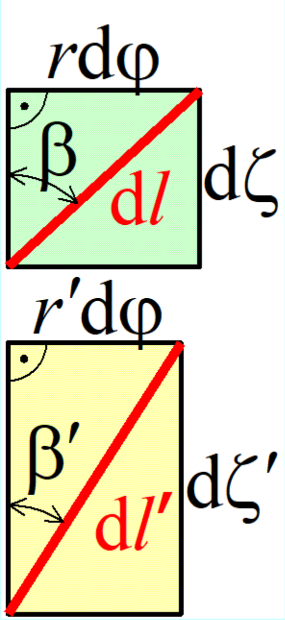
Radial strain: $$\varepsilon_r = \frac{\mathrm{d}r' - \mathrm{d}r}{\mathrm{d}r} = \frac{\mathrm{d}r'}{\mathrm{d}\zeta} - 1, \quad \mathrm{d}r' = (1+\varepsilon_r)\,\mathrm{d}r$$

Contraction ratio (like Poisson): $$\eta = -\varepsilon_r / \varepsilon_a$$

Fiber strain: $$\varepsilon_l = \frac{\mathrm{d}l' - \mathrm{d}l}{\mathrm{d}l} = \frac{\mathrm{d}l'}{\mathrm{d}l} - 1, \quad \frac{\mathrm{d}l'}{\mathrm{d}l} = 1 + \varepsilon_l$$



Based on the Pythagorean theorem, it is valid

- before deformation: $\mathrm{d}^2 l = \mathrm{d}^2\zeta + (r\,\mathrm{d}\varphi)^2$
- after deformation: $\mathrm{d}^2 l' = \overbrace{\mathrm{d}^2\zeta'}^{=(1+\varepsilon_a)^2\mathrm{d}^2\zeta} + \overbrace{(r'\mathrm{d}\varphi)^2}^{=(1+\varepsilon_r)^2 r'^2 \mathrm{d}^2\varphi} =$

$$= (1+\varepsilon_a)^2\,\mathrm{d}^2\zeta + \Big(1 + \overbrace{\varepsilon_r}^{=-\eta\varepsilon_a}\Big)^2 (r\,\mathrm{d}\varphi)^2 = (1+\varepsilon_a)^2\,\mathrm{d}^2\zeta + (1-\eta\varepsilon_a)^2 (r\,\mathrm{d}\varphi)^2$$

Note: Because of continuity of yarn body, $\mathrm{d}\varphi$ must be the same.

Using earlier derived equation, we obtain

$$(1+\varepsilon_l)^2=\frac{d^2l'}{d^2l}=\frac{(1+\varepsilon_a)^2d^2\zeta+(1-\eta\varepsilon_a)^2(rd\varphi)^2}{d^2\zeta+(r\,d\varphi)^2}=\frac{(1+\varepsilon_a)^2+(1-\eta\varepsilon_a)^2\Big(\overbrace{rd\varphi/d\zeta}^{=\tan\beta}\Big)^2}{1+\Big(\overbrace{rd\varphi/d\zeta}^{=\tan\beta}\Big)^2}=$$

$$=\frac{(1+\varepsilon_a)^2+(1-\eta\varepsilon_a)^2\tan^2\beta}{1+\tan^2\beta}=\frac{\left(1+2\varepsilon_a+\varepsilon_a^2\right)+\left(1-2\eta\varepsilon_a+\eta^2\varepsilon_a^2\right)\tan^2\beta}{1+\tan^2\beta}=$$

$$=\frac{1+2\varepsilon_a+\varepsilon_a^2+\tan^2\beta-2\eta\varepsilon_a\tan^2\beta+\eta^2\varepsilon_a^2\tan^2\beta}{1+\tan^2\beta}=1+\frac{2\varepsilon_a-2\eta\varepsilon_a\tan^2\beta}{\underbrace{1+\tan^2\beta}_{1/\cos^2\beta}}+\frac{\varepsilon_a^2+\eta^2\varepsilon_a^2\tan^2\beta}{\underbrace{1+\tan^2\beta}_{1/\cos^2\beta}}$$

$$(1+\varepsilon_l)^2=1+2\varepsilon_a\left(\cos^2\beta-\eta\sin^2\beta\right)+\varepsilon_a^2\left(\cos^2\beta+\eta^2\sin^2\beta\right)$$

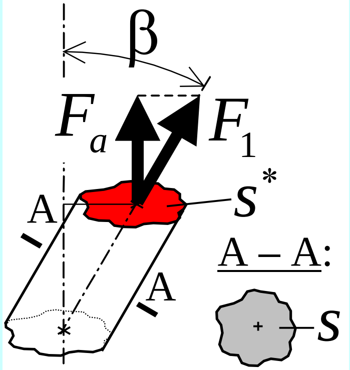
Assumption: Strains are small. Then $\varepsilon_l^2\to 0,\ \varepsilon_a^2\to 0$ and

$$1+2\varepsilon_l+\underbrace{\varepsilon_l^2}_{\to 0}=1+2\varepsilon_a\left(\cos^2\beta-\eta\sin^2\beta\right)+\underbrace{\varepsilon_a^2}_{\to 0}\left(\cos^2\beta+\eta^2\sin^2\beta\right)$$

$$\varepsilon_l=\varepsilon_a\left(\cos^2\beta-\eta\sin^2\beta\right)$$

Note: Gegauff (1907) used $\eta=0$ and then he obtained $\varepsilon_l=\varepsilon_a\cos^2\beta$

Assumption (easiest case): <u>The fiber tensile stress-strain relation is linear</u> $\sigma = E\varepsilon_l$, where σ...tensile stress and E...Young modulus. Axial force in fiber is $F_1 = \sigma s = E\varepsilon_l s$ and the component force in the direction of yarn axis is $F_a = F_1 \cos\beta = E\varepsilon_l s \cos\beta$. The fiber sectional area (red) is $s^* = s/\cos\beta$. Normal stress on this area is

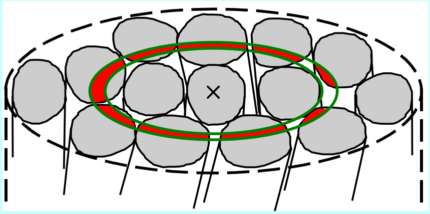


$$\sigma_a = \frac{F_a}{s^*} = \frac{E\varepsilon_l s\cos\beta}{s/\cos\beta} = E \overbrace{\varepsilon_l}^{\varepsilon_a\left(\cos^2\beta - \eta\sin^2\beta\right)} \cos^2\beta$$

$$\sigma_a = E\varepsilon_a\left(\cos^4\beta - \eta\sin^2\beta\cos^2\beta\right)$$

(Small deformation is assumed.)

Fiber sectional area inside the differential annulus (red) is $\mathrm{d}S = 2\pi r\,\mathrm{d}r\,\mu$ (derived earlier) and then the <u>yarn axial force</u> is



$$P = \int_{r=0}^{r=D/2} \sigma_a \mathrm{d}S$$

Rearrangement:

$$P = \int_{r=0}^{r=D/2} \sigma_a \mathrm{d}S = \int_0^{D/2} \overbrace{E\varepsilon_a\left(\cos^4\beta - \eta\sin^2\beta\cos^2\beta\right)}^{=\sigma_a} \overbrace{2\pi r\,\mathrm{d}r\,\mu}^{=\mathrm{d}S} =$$

$$= 2\pi\mu E\varepsilon_a \int_0^{D/2}\left(\cos^4\beta - \eta\sin^2\beta\cos^2\beta\right) r\,\mathrm{d}r =$$

$$\text{Substitution: } r = \frac{\overbrace{2\pi r Z}^{=\tan\beta}}{2\pi Z} = \frac{D\tan\beta}{2\underbrace{\pi D Z}_{=\tan\beta_D}} = \frac{D\tan\beta}{2\tan\beta_D}, \quad \mathrm{d}r = \frac{D}{2\tan\beta_D}\frac{\mathrm{d}\beta}{\cos^2\beta},$$

$$r\,\mathrm{d}r = \overbrace{\left(\frac{D\tan\beta}{2\tan\beta_D}\right)}^{=r}\overbrace{\left(\frac{D}{2\tan\beta_D}\frac{\mathrm{d}\beta}{\cos^2\beta}\right)}^{=\mathrm{d}r} = \left(\frac{D}{2\tan\beta_D}\right)^2\frac{\sin\beta}{\cos^3\beta}\mathrm{d}\beta$$

$$= 2\pi\mu E\varepsilon_a \int_{\beta=0}^{\beta=\beta_D}\left(\cos^4\beta - \eta\sin^2\beta\cos^2\beta\right)\left(\frac{D}{2\tan\beta_D}\right)^2\frac{\sin\beta}{\cos^3\beta}\mathrm{d}\beta =$$

$$= 2\pi\mu E\varepsilon_a\left(\frac{D}{2\tan\beta_D}\right)^2\int_0^{\beta_D}\left(\cos^4\beta - \eta\sin^2\beta\cos^2\beta\right)\frac{\sin\beta}{\cos^3\beta}\mathrm{d}\beta$$

The contraction ratio η is generally a function of radius r of element, yarn twist intensity, etc. But usually it is *assumed:* contraction ratio $\eta = \mathrm{const.}$ is a yarn parameter.

Then $P = 2\pi\mu E\varepsilon_a \left[D/(2\tan\beta_D)\right]^2 \left[\int_0^{\beta_D} \sin\beta\cos\beta\,\mathrm{d}\beta - \eta\int_0^{\beta_D}\left(\sin^3\beta/\cos\beta\right)\mathrm{d}\beta\right]$

Indefinite integrals:

$$\int \sin\beta\cos\beta\,\mathrm{d}\beta = \int t\,\mathrm{d}t = t^2/2 = \sin^2\beta/2 = \left(1-\cos^2\beta\right)/2;$$

Substitution: $\sin\beta = t,\ \cos\beta\,\mathrm{d}\beta = \mathrm{d}t$

$$\int \frac{\sin^3\beta}{\cos\beta}\mathrm{d}\beta = \int \frac{\left(1-\cos^2\beta\right)}{\cos\beta}\sin\beta\,\mathrm{d}\beta = \int \frac{\left(1-t^2\right)}{t}(-\mathrm{d}t) = -\int\frac{\mathrm{d}t}{t} + \int t\,\mathrm{d}t =$$

Substitution: $\cos\beta = t,\ -\sin\beta\,\mathrm{d}\beta = \mathrm{d}t$

$$= -\ln|t| + t^2/2 = -\ln|\cos\beta| + \cos^2\beta/2;$$

$$\int\left(\sin\beta\cos\beta - \eta\frac{\sin^3\beta}{\cos\beta}\right)\mathrm{d}\beta = \sin^2\beta/2 = \frac{\left(1-\cos^2\beta\right)}{2} + \eta\ln|\cos\beta| - \eta\frac{\cos^2\beta}{2} =$$

$$= \frac{1}{2}\left[1 + \eta - \eta - \cos^2\beta + \eta\underbrace{2\ln|\cos\beta|}_{=\ln\cos^2\beta} - \eta\cos^2\beta\right] = \frac{1}{2}\left[-\eta + (1+\eta)\left(1-\cos^2\beta\right) + \eta\ln\cos^2\beta\right]$$

Yarn axial force is now

$$P = 2\pi\mu E\varepsilon_a \left(\frac{D}{2}\right)^2 \frac{1}{\tan^2\beta_D}\left\{\frac{1}{2}\left[-\eta+(1+\eta)\left(\overbrace{1-\cos^2\beta_D}^{=\sin^2\beta_D}\right)+\eta\ln\cos^2\beta_D\right]-\right.$$

$$\left.-\frac{1}{2}\left[-\eta+(1+\eta)\left(\overbrace{1-\cos^2 0}^{=0}\right)+\eta\,\overbrace{\ln\cos^2 0}^{=0}\right]\right\}$$

$$P = \pi\mu E\varepsilon_a (D/2)^2\left[(1+\eta)\cos^2\beta_D + \eta\left(\ln\cos^2\beta_D\right)/\tan^2\beta_D\right]$$

At the same strain ε_a, untwisted fiber bundle of the same count (fineness) has same substantial cross-sectional area and the axial force is $P^* = \overbrace{\sigma_a}^{=E\varepsilon_a}\ \overbrace{S}^{=\mu\,\pi D^2/4} = \pi\mu E\varepsilon_a (D/2)^2$. **Tensile force utilization coefficient** in the twisted yarn is then

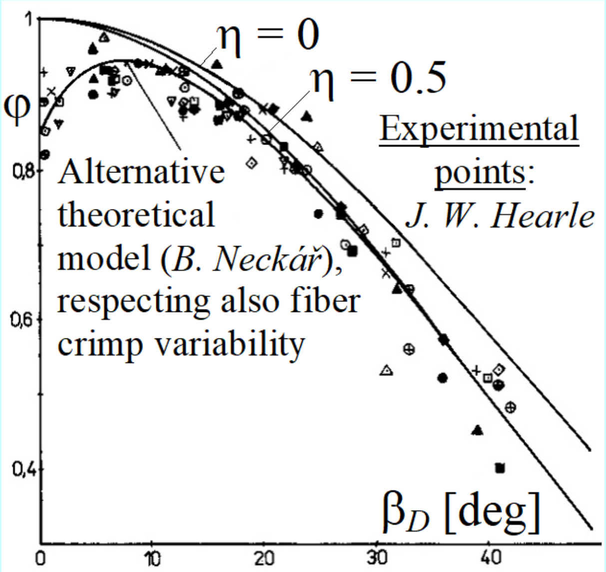
$$\varphi = P/P^* = (1+\eta)\cos^2\beta_D + \eta\left(\ln\cos^2\beta_D\right)/\tan^2\beta_D$$

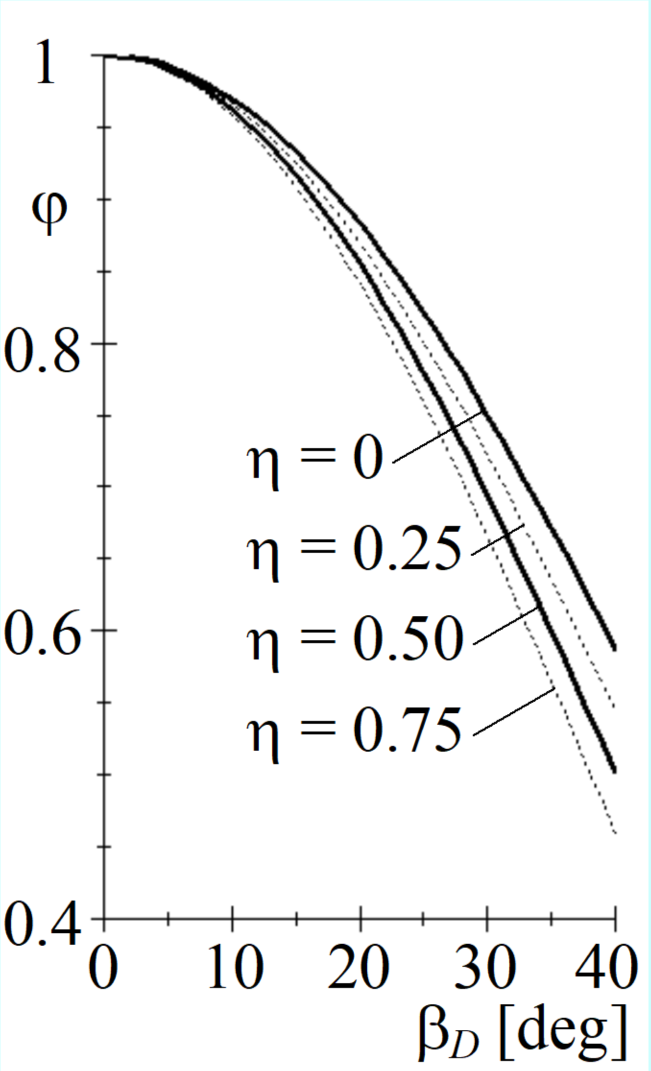
Graphical representation of the last equation is shown on the figure:

Notes:

1. Using $\eta = 0$ we get $\varphi = \cos^2 \beta_D$, derived by *Gegauff (1907)*

2. φ is applicable also for determination of filament yarn strength (by $\eta = 0.5$) – see example

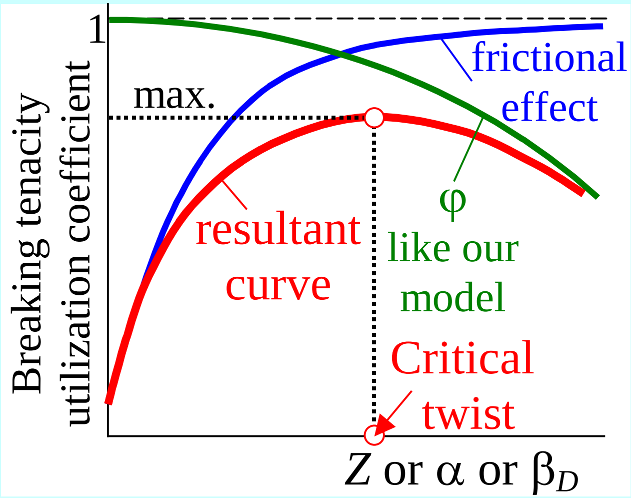




3. <u>Strength of staple yarn</u> is usually interpreted – beside others - as

a) a resultant of fiber path and fiber straining in yarn and

b) a complex of frictional mechanisms.



First of them (|) is described – say about - by our φ (in an easiest case). The second one (|) is still an open problem in yarn theory.

Closing note: Helical model is the best known theoretical concept in internal yarn geometry. We showed only a few basic imaginations in this lecture. Many other versions and their applications can be found in traditional textile literatures.