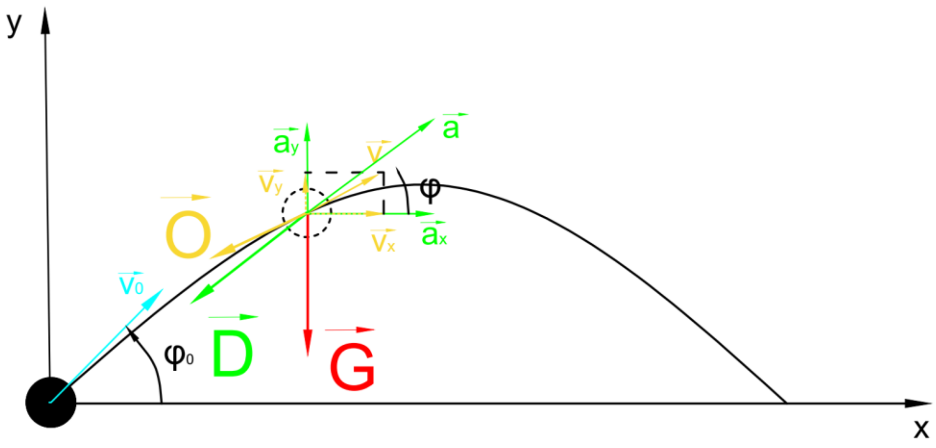
Curvilinear motion

Throw a particle into space with the initial angle of elevation α and initial velocity v_0. The mass of the particle is m. When the particle moves with the drag force given by the formula: $\vec{O} = -H\vec{v}$.

Task: Find the x and y motion of the particle as functions of time x(t), y(t)

Solution

First, we draw the free body diagram as the figure.



The 2d cartesian coordinate system is chosen and placed at the initial position of the particle.

The forces act on the particle are:

the drag force: $\vec{O} = -H\vec{v}$, this force against the direction of motion.

the gravity force: $\vec{G} = m\vec{g}$

and the D'Alembert force $\vec{D}$

Generally, **the equation of motion** can be written in accodence with the D'Alembert's law:

$$\sum \vec{F} + \vec{D} = \vec{0}$$

Or:

$$m\vec{a} = \vec{G} + \vec{O}$$

Then we can write the component equations:

(x): $m\ddot{x} = -O.\cos\varphi$

(y): $m\ddot{y} = -O.\sin\varphi - mg$

Or:

$m\ddot{x} = -Hv.\cos\varphi = -H\dot{x}$

$m\ddot{y} = -Hv.\sin\varphi - mg = -H\dot{y} - mg$

Then we get:

$\ddot{x} + \frac{H}{m}\dot{x} = 0\,(1)$

$\ddot{y} + \frac{H}{m}\dot{y} = -g\,(2)$

Solve the homogeneous equation (1):

We are easy to see the auxiliary equation of equation (1) as follows:

$\lambda^2 + \frac{H}{m}\lambda = 0\,(3)$

The equation (3) has two solutions:

$\lambda_1 = 0$

$\lambda_2 = -\frac{H}{m}$ (4)

Then we can get the general solutions of equation (1) in the form:

$x_H = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

Consider to (4), we have:

$x = x_H = C_1 + C_2 e^{-\frac{H}{m}t}$ (5)

Solve the nonhomogeneous equation (2):

In the same way, we have the solutions of the homogenous equation $\ddot{y} + \frac{H}{m}\dot{y} = 0$ (6) in the form:

$y_H = C_3 + C_4 e^{-\frac{H}{m}t}$ (7)

Now we are going to find particular solution of equation (2)

We assume that a particular solution can be written in the form:

$y_H = At$ (8)

where A is a constant.

So we get:

$$\dot{y}_H = A(9)$$

And

$$\ddot{y}_H = 0(10)$$

Substitute (9) and (10) into (2), we have:

$$0 + \frac{H}{m}A = -g$$

So we find:

$$A = -\frac{mg}{H}$$

Then we obtain the general solution of equation (2) in the form:

$$y = y_H + y_p = C_3 + C_4 e^{-\frac{H}{m}t} - \frac{mg}{H}t\,(11)$$

Consider the initial conditions:

$$\begin{aligned} &t = t_0 = 0 \\ &x(0) = 0,\ y(0) = 0 \qquad (12) \\ &\dot{x}(0) = v_0 \cos\alpha,\ \dot{y}(0) = v_0 \sin\alpha \end{aligned}$$

We are going to find the coefficients C_1, C_2, C_3, C_4

From (5), we have:

$$\dot{x} = -\frac{H}{m}C_2 e^{-\frac{H}{m}t}\,(13)$$

Combine (5), (13) and (12), we have:

$$x(0) = C_1 + C_2 = 0$$

$$\dot{x}(0) = -\frac{H}{m}C_2 = v_0 \cos\alpha$$

So :

$$C_1 = -C_2$$

$$C_2 = -\frac{m}{H}v_0 \cos\alpha$$

From (11), we have:

$$\dot{y} = C_3 - \frac{H}{m} C_4 e^{-\frac{H}{m}t} - \frac{mg}{H} \quad (14)$$

Combine (11), (14) and (12), we have:

$$y(0) = C_3 + C_4 = 0$$

$$\dot{y}\left(0\right) = -\frac{H}{m} C_4 - \frac{mg}{H} = v_0 \sin\alpha$$

So :

$$C_3 = -C_4$$

$$C_4 = -\frac{m}{H} v_0 \sin\alpha - \frac{m^2 g}{H}$$