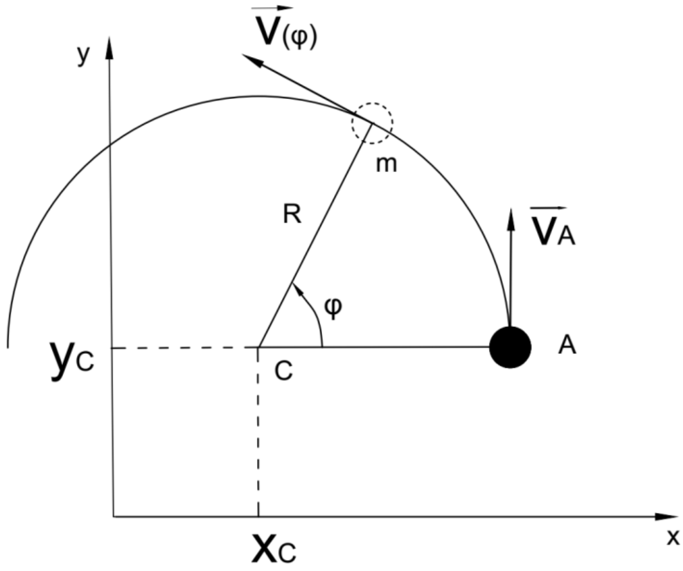
Dynamics of particle in potential force field

Given: A particle (mass m) moves on the trajectory which is a circle (with the radius R, the center C(x_C, y_C)) as the Figure. A force field acting on the particle is given by vector expression: $\vec{F} = F_x\vec{i} + F_y\vec{j}$, in which: $F_x = -F_0R\left(\frac{y}{x^2}+\frac{1}{y}\right), F_y = F_0R\left(\frac{y}{x^2}+\frac{1}{x}\right)$ and F_0, x_C, y_C are given by constants. The initial velocity of the particle is $v(0)=v_A$.



Task:

1, Is F potential force or not?

2, If F is potential force, find expression of potential energy $V(x,y)$=?

3, Find the expression of the velocity of the particle $v(x)$=?

Solution:

1, To answer the question, we are going to check the condition of potential force field:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

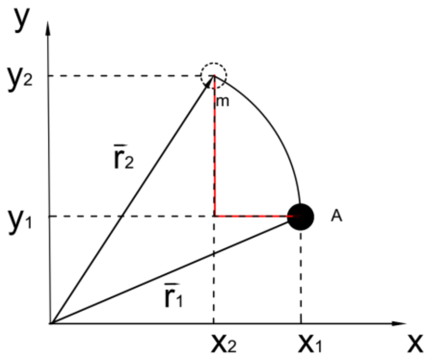
We have:

$$F_x = -F_0R\left(\frac{y}{x^2}+\frac{1}{y}\right) \to \frac{\partial F_x}{\partial y} = -F_0R\left(\frac{1}{x^2}-\frac{1}{y^2}\right) \text{(1)}$$

$$F_x = F_0R\left(\frac{x}{y^2}+\frac{1}{x}\right) \to \frac{\partial F_y}{\partial x} = F_0R\left(-\frac{1}{x^2}+\frac{1}{y^2}\right) \text{ (2)}$$

It is easy to see $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$. So the force F is a potential force.

2, Because F is a potential force so the work only depends on the starting point and the ending point but it does not depends on the trajectory. Then we can choose the trajectory as the figure (in green) for convenience.



So we have:

$$\int_{\vec r_1}^{\vec r_2} \vec F d\vec r = \int_{x_1}^{x_2} F_x(x,y_1)dx + \int_{y_1}^{y_2} F_y(x_2,y)dy$$

$$= -F_0R\int_{x_1}^{x_2}\left(\frac{y_1}{x^2}+\frac{1}{y_1}\right)dx + F_0R\int_{y_1}^{y_2}\left(\frac{x_2}{y_2}+\frac{1}{x}\right)dy$$

$$= -F_0R\left(-\frac{y_1}{x}+\frac{x}{y_1}\right)\Big|_{x_1}^{x_2} + F_0R\left(-\frac{x_2}{y}+\frac{y}{x_2}\right)\Big|_{y_1}^{y_2}$$

$$= F_0R\left(\frac{y_1}{x_2}-\frac{x_2}{y_1}-\frac{y_1}{x_1}+\frac{x_1}{y_1}-\frac{x_2}{y_2}+\frac{y_2}{x_2}+\frac{x_2}{y_1}-\frac{y_1}{x_2}\right)$$

$$= -F_0R\left(\frac{x_2}{y_2}-\frac{y_2}{x_2}\right) + F_0R\left(\frac{x_1}{y_1}-\frac{y_1}{x_1}\right)$$

$$= -V(x_2,y_2)+V(x_1,y_1)$$

Where:

$$V(x,y) = F_0R\left(\frac{x}{y}-\frac{y}{x}\right)$$

V is the expression of potential energy.

3, Accordance with the principle of conservation of energy, we have:

$$K_A + V_A = K + V$$

Or:

$$\frac{1}{2}mv_A^2 + V(x_A,y_A) = \frac{1}{2}mv^2 + V(x,y)$$

$$\rightarrow v = \sqrt{v_A^2 + \frac{2}{m}\left(V(x,y)-V(x_A,y_A)\right)}$$

In which:

$$x_A = x_C + R,\ x = x_C + R\cos\varphi$$
$$y_A = y_C \qquad ,\ y = y_C + R\sin\varphi$$

So:

$$v = \sqrt{v_A^2 + \frac{2F_0R}{m}\left(\frac{x_C + R}{y_C} - \frac{y_C}{x_C + R} - \frac{x_C + R\cos\varphi}{y_C + R\sin\varphi} + \frac{y_C + R\sin\varphi}{x_C + R\cos\varphi}\right)}$$