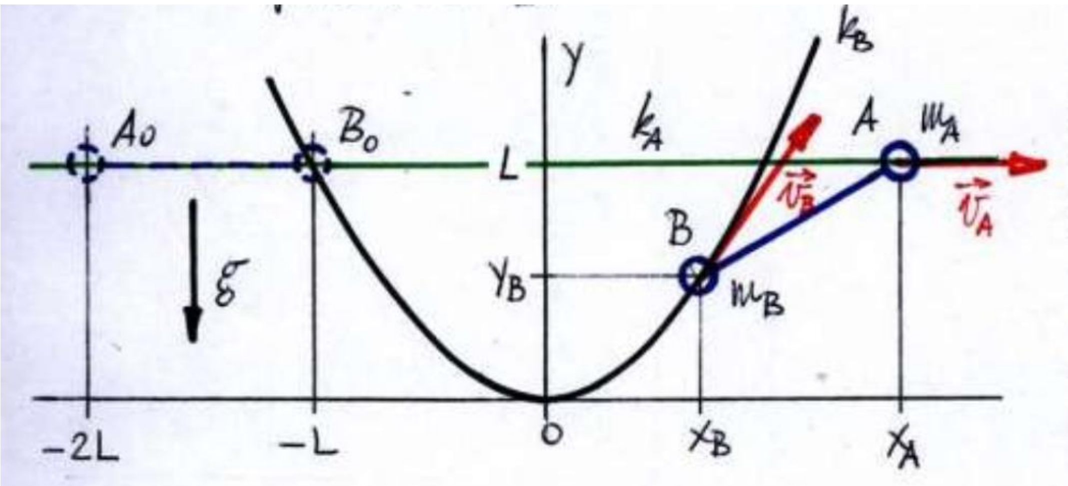
Law of conservation of mechanical energy of a system of two particles

The system of two particles A, B with a given mass. They are connected by a bar (without mass and with a given length). When the system moves, the trajectory of particle A is the line k_A and the trajectory of particle B is the parabola k_B (as shown in the Figure)



Given:

the mass of particles: m_A, m_B,

the length of the bar: L,

the trajectory of particle A, k_A: $y_A = L$,

the trajectory of particle B, k_B: $y_B = \frac{x_B^2}{L}$,

initial position: $x_B = -L, x_A = -2L$,

potential energy for gravity force acting on the particles can be computed by:

$V(y) = mgy$

Task:

Find the function of the velocity of the particles $v_A(x_A), v_B(x_B)$

Find the function of displacement $x_B(t)$

Solution:

Because the working force acting on the system is a potential force, we can use the law of change of mechanical energy

$$K_0 + V_0 = K + V \tag{1}$$

With the assumption that the initial velocity of the particles are 0, $v_{A_0} = v_{B_0} = 0$

So the initial kinetic energy $K_0 = 0$

And the potential energy is given by:

$$V_0 = m_A gL + m_B gL$$

The general kinetic energy and general potential energy are given by:

$$K = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

And

$$V = m_A gL + m_B g y_B$$

Substitute to (1), we have

$$m_A gL + m_B gL = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 + m_A gL + m_B g y_B$$

Simplify and get:

$$m_A v_A^2 + m_B v_B^2 = 2 m_B \left(L - y_B \right) \quad (2)$$

Because the distance between particle A and particle B is L (the length of the bar), so we can write:

$$L^2 = \left(x_A - x_B \right)^2 + \left(y_A - y_B \right)^2 \quad (3)$$

From the function of the trajectories of the particles, we have:

$$y_A = L$$

$$y_B = \frac{x_B^2}{L} \quad (4)$$

Substitute (4) to (3), we get:

$$L^2 = \left(x_A - x_B \right)^2 + \left(L - \frac{x_B^2}{2} \right)^2$$

$$L^2 = x_A^2 - 2 x_A x_B + x_B^2 + L^2 - 2L \frac{x_B^2}{L} + \frac{x_B^4}{L^2}$$

$$\rightarrow 0 = x_A^2 - 2 x_A x_B - x_B^2 + \frac{x_B^4}{L^2}$$

The equation above is a quadratic equation with the variable x_A and the roots can be presented in the form:

$$x_{A1,2} = x_B \pm \sqrt{x_B^2 + x_B^2 - \frac{x_B^4}{L^2}}$$

Simplify, we get:

$$x_{A_{1,2}} = x_B\left[1 \pm \sqrt{2 - \left(\frac{x_B}{L}\right)^2}\right] \tag{5}$$

The velocity of the particles are:

$$v_A = \dot{x}_A$$
$$v_B = \sqrt{\dot{x}_B^2 + \dot{y}_B^2}$$

So :

$$v_A^2 = \dot{x}_A^2 \tag{6}$$
$$v_B^2 = \dot{x}_B^2 + \dot{y}_B^2 \tag{7}$$

Make the derivation of (4):

$$\dot{y}_B = 2\frac{x_B}{L}\dot{x}_B$$

Then substitute to (7), we get:

$$v_B^2 = \dot{x}_B^2\left[1 + 4\left(\frac{x_B}{L}\right)^2\right] \tag{8}$$

Make the derivation of (5), we have:

$$\dot{x}_A = \dot{x}_B\left[1 \pm \sqrt{2 - \left(\frac{x_B}{L}\right)^2}\right] + x_B\frac{-\frac{x_B}{L}\frac{\dot{x}_B}{L}}{\sqrt{2 - \left(\frac{x_B}{L}\right)^2}}$$

$$\dot{x}_A = \dot{x}_B\left[1 \pm \sqrt{2 - \left(\frac{x_B}{L}\right)^2} - \frac{\left(\frac{x_B}{L}\right)^2}{\sqrt{2 - \left(\frac{x_B}{L}\right)^2}}\right]$$

The substitute to (6), we have:

$$v_A^2 = \dot{x}_B^2 \left[1 \pm \sqrt{2 - \left(\frac{x_B}{L}\right)^2} - \frac{\left(\frac{x_B}{L}\right)^2}{\sqrt{2 - \left(\frac{x_B}{L}\right)^2}} \right]^2 \tag{9}$$

Substitute (8) and (9) to (7), we have:

$$v_B^2 = v_A^2 \frac{1 + 4\left(\frac{x_B}{L}\right)^2}{\left[1 \pm \sqrt{2 - \left(\frac{x_B}{L}\right)^2} - \frac{\left(\frac{x_B}{L}\right)^2}{\sqrt{2 - \left(\frac{x_B}{L}\right)^2}} \right]^2} \tag{10}$$

So we can rewrite (2):

$$v_A^2 \left\{ m_A + \frac{m_B \left[1 + 4\left(\frac{x_B}{L}\right)^2 \right]}{\left[1 \pm \sqrt{2 - \left(\frac{x_B}{L}\right)^2} - \frac{\left(\frac{x_B}{L}\right)^2}{\sqrt{2 - \left(\frac{x_B}{L}\right)^2}} \right]^2} \right\} = 2 m_B g L \left[1 - \left(\frac{x_B}{L}\right)^2 \right]$$

So:

$$v_A^2 = \frac{2 m_B g L \left[1 - \left(\frac{x_B}{L}\right)^2 \right]}{m_A + \frac{m_B \left[1 + 4\left(\frac{x_B}{L}\right)^2 \right]}{\left[1 \pm \sqrt{2 - \left(\frac{x_B}{L}\right)^2} - \frac{\left(\frac{x_B}{L}\right)^2}{\sqrt{2 - \left(\frac{x_B}{L}\right)^2}} \right]^2}}$$

$$v_B^2 = \frac{2m_B gL\left[1-\left(\frac{x_B}{L}\right)^2\right]\left[1+4\left(\frac{x_B}{L}\right)^2\right]}{m_A\left[1\pm\sqrt{2-\left(\frac{x_B}{L}\right)^2}-\frac{\left(\frac{x_B}{L}\right)^2}{\sqrt{2-\left(\frac{x_B}{L}\right)^2}}\right]^2+m_B\left[1+4\left(\frac{x_B}{L}\right)^2\right]} \qquad (11)$$

From (8), we have:

$$\dot{x}_B = \frac{dx_B}{dt} = \sqrt{\frac{v_B^2}{1+4\left(\frac{x_B}{L}\right)^2}}$$

$$\to \sqrt{\frac{1+4\left(\frac{x_B}{L}\right)^2}{v_B^2}}\,dx_B = dt$$

$$\to \int_{-L}^{x_B}\sqrt{\frac{m_A\left[1\pm\sqrt{2-\left(\frac{x_B}{L}\right)^2}-\frac{\left(\frac{x_B}{L}\right)^2}{\sqrt{2-\left(\frac{x_B}{L}\right)^2}}\right]^2+m_B\left[1+4\left(\frac{x_B}{L}\right)^2\right]}{2m_B gL\left[1-\left(\frac{x_B}{L}\right)^2\right]}}\,dx_B = \int_0^t dt$$

Solve the integral equation above we obtain the $x_B = x_B(t)$