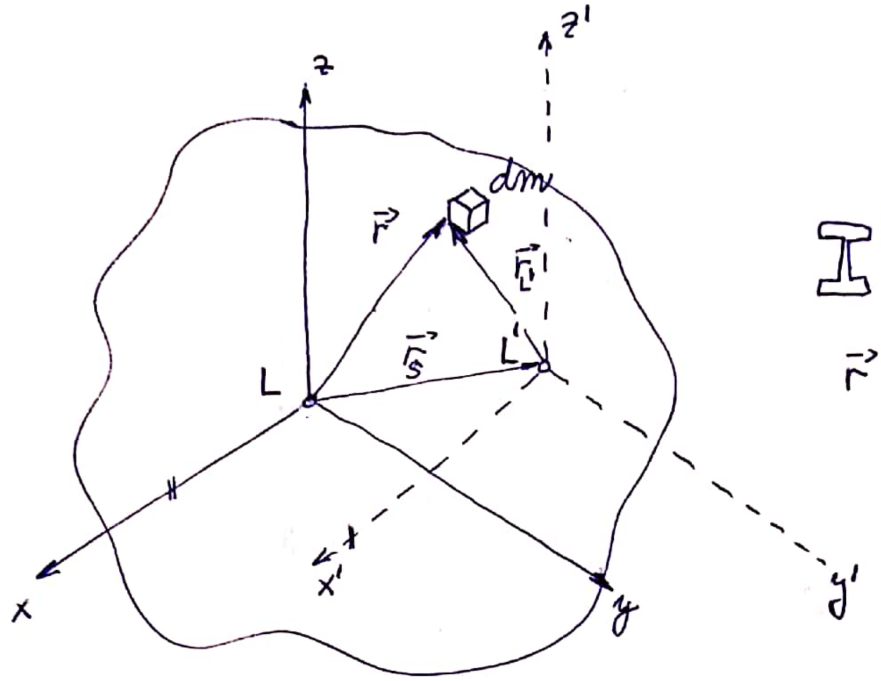
MOMENTS OF INERTIA WITH RESPECT TO SHFTED AXES



$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r}_s = x_s\vec{i} + y_s\vec{j} + z_s\vec{k}$$

$$I^{(L,x,y,z)} \rightarrow I^{(L',x',y',z')}$$

$$\vec{r} = \vec{r}_s + \vec{r}_{L'}$$

$$\Downarrow$$

$$\vec{r}_{L'} = \vec{r} - \vec{r}_s$$

moment of inertia of body with respect to origin of coordinate system:

$$J_L' = \int_{(m)} (\vec{r}_{L'})^2 \, dm = \int_{(m)} (\vec{r} - \vec{r}_s) \cdot (\vec{r} - \vec{r}_s) \, dm =$$

$$= \int_{(m)} (r^2 - 2\vec{r}_s \cdot \vec{r} + r_s^2) \, dm$$

$$\vec{r}_s \cdot \vec{r} = (x_s, y_s, z_s) \cdot (x, y, z) = x \cdot x_s + y\, y_s + z\, z_s$$

we can write:

$$J_{y'z'} = \int_{(m)} (x - x_s)(x - x_s) \, dm = \int_{(m)} (x^2 - 2x x_s + x_s^2) \, dm =$$

$$= J_{yz} - 2x_s S_{yz} + x_s^2 m$$

$$J_{x'y'} = J_{xy} - 2z_s S_{xy} + z_s^2 m$$

$$J_{x'z'} = J_{xz} - 2y_s S_{xz} + y_s^2 m$$

$$J_{x'} = J_{x'y'} + J_{x'z'} = J_{xy} - 2 z_s \cdot S_{xy} + z_s^2 m + J_{xz} - 2 y_s S_{xz} + y_s^2 m =$$

$$= J_{xy} + J_{xz} - 2(z_s S_{xy} + y_s S_{xz}) + (y_s^2 + z_s^2) m =$$

$$= J_x - 2(y_s S_{xz} + z_s S_{xy}) + (y_s^2 + z_s^2) m$$

$$J_{y'} = J_{y'x'} + J_{y'z'} = J_y - 2(x_s S_{yz} + z_s S_{xy}) + (x_s^2 + z_s^2) \cdot m$$

$$J_{z'} = J_{x'z'} + J_{y'z'} = J_z - 2(x_s^2 S_{yz} + y_s S_{xz}) + (x_s^2 + y_s^2) m$$

PRODUCTS OF INERTIA

$$D_{x'y'} = \int_{(m)} (x - x_s)(y - y_s)\, dm = \int_{(m)} xy\, dm - x_s \int_{(m)} y\, dm - y_s \int_{(m)} x\, dm + x_s y_s\, dm =$$

$$= D_{xy} - x_s S_{xz} - y_s S_{yz} + x_s y_s m$$

$$D_{y'z'} = D_{yz} - y_s S_{xy} - z_s S_{xz} + y_s z_s m$$

$$D_{x'z'} = D_{xz} - x_s S_{xy} - z_s S_{yz} + x_s z_s \cdot m$$

SIMLIFICATION: IF $L \equiv C$ (center of mass) $\Rightarrow$

$\Rightarrow \vec{S} = \vec{0} \Rightarrow S_{xy} = S_{xz} = S_{yz} = 0$

then:

$$J_{L'} = J_L + r_s^2 \cdot m$$

$$J_{x'y'} = J_{xy} + z_s^2 m$$

$$J_{y'z'} = J_{yz} + x_s^2 m$$

$$J_{x'z'} = J_{xz} + y_s^2 m$$

$$J_{x'} = J_x + (y_s^2 + z_s^2) m$$

$$J_{y'} = J_y + (x_s^2 + z_s^2) m$$

$$J_{z'} = J_z + (x_s^2 + y_s^2) m$$

$$D_{x'y'} = D_{xy} + x_s y_s m$$

$$D_{y'z'} = D_{yz} + y_s z_s m$$

$$D_{x'z'} = D_{xz} + x_s z_s m$$

Steiner's theorem

MATRIX OF INERTIA

$$\mathbb{I}' = \mathbb{I} + m \begin{bmatrix} y_s^2 + z_s^2 & -x_s y_s & -x_s z_s \\ -x_s y_s & x_s^2 + z_s^2 & -y_s z_s \\ -x_s z_s & -y_s \cdot z_s & x_s^2 + y_s^2 \end{bmatrix}$$