Dynamics of rotational motion of a body

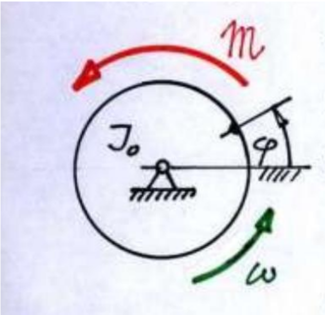
The rotor of an electric motor with a given moment of inertia to the axis of rotation. It works under the driving torque characterized by the given moment characteristic.

Given:

- Moment of inertia J_0;
- The given moment characteristic:

$$M = M_0\left(1-\frac{\omega}{\omega_0}\right)\left[8\left(\frac{\omega}{\omega_0}\right)^2+\frac{\omega}{\omega_0}+1\right]$$

- The constants M_0, ω_0 ;
- The initial condition: $\omega(0)=0$



Task:

- Determine the dependence of the angular velocity ω of the rotor on the position angle φ

Solution:

Because the rotor is dynamically balanced so we just need to concern to d'Alembert moment in the equation below:

$$M + M_D = 0$$

Where: $M_D = -J_0\alpha$

So we have:

$$J_0\alpha = M \tag{1}$$

The angular acceleration can be rewritten in the form:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\varphi}.\frac{d\varphi}{dt} = \omega\frac{d\omega}{d\varphi} \tag{2}$$

Substitute (2) to (1) and combine with the expression of given applied moment, we have:

$$J_0\omega\frac{d\omega}{d\varphi} = M_0\left(1-\frac{\omega}{\omega_0}\right)\left[8\left(\frac{\omega}{\omega_0}\right)^2+\frac{\omega}{\omega_0}+1\right] \tag{3}$$

For convenient of solving, we denote $x = \frac{\omega}{\omega_0}$;

Then we can rewrite the equation (3) as follows:

$$J_0\omega_0^2 \frac{dx}{d\varphi} = M_0(1-x)\left[8x^2 + x + 1\right] \tag{4}$$

Separate the variables, we obtain:

$$\frac{xdx}{(1-x)\left(8x^2+x+1\right)} = \frac{M_0}{J_0\omega_0^2} d\varphi \tag{5}$$

When $\omega(0)=0$, we have $x(0)=0$ proportionally.

So we make the integration:

$$\int_0^{x(\varphi)} \frac{xdx}{(1-x)\left(8x^2+x+1\right)} = \frac{M_0}{J_0\omega_0^2}\int_0^{\varphi} d\varphi \tag{6}$$

Assume that we can make the realization:

$$\frac{x}{(1-x)\left(8x^2+x+1\right)} = \frac{A}{1-x} + \frac{Bx+C}{8x^2+x+1} \tag{7}$$

So:

$$x = A\left(8x^2+x+1\right) + (Bx+C)(1-x)$$

$$x = A8x^2 + Ax + A + Bx + C - Bx^2 - Cx$$

$$(8A-B)x^2 + (A+B-C)x + (A+C) = x$$

So it is easy to get:

$$8A - B = 0$$

$$A + B - C = 1$$

$$A + C = 0$$

We get:

$$A = \frac{1}{10}, B = \frac{8}{10}, C = -\frac{1}{10}$$

Let’s consider the left side of (6), we make the analyzation as follow:

$$\int_0^x \frac{xdx}{(1-x)\left(8x^2+x+1\right)} = \int_0^x \frac{A}{1-x}dx + \int_0^x \frac{Bx+C}{8x^2+x+1}dx \tag{8}$$

$$\int_0^x \frac{A}{1-x}dx = -\frac{1}{10}\int_0^x \frac{1}{1-x}dx = -\frac{1}{10}\ln|x-1| \tag{9}$$

$$\int_0^x \frac{Bx+C}{8x^2+x+1}dx = \int_0^x \left[\frac{B}{16}\frac{16x+1}{8x^2+x+1} + \left(C - \frac{B}{16}\right)\frac{1}{8x^2+x+1}\right]dx \tag{10}$$

$$= \frac{1}{20}\int_0^x \frac{16x+1}{8x^2+x+1}dx - \frac{3}{20}\int_0^x \frac{1}{8x^2+x+1}dx$$

$$= \frac{1}{20}\ln\left(8x^2+x+1\right) - \frac{3}{20}\frac{2}{\sqrt{4.8-1}}\left[arctg\frac{16x+1}{\sqrt{4.8-1}}\right]_0^x$$

$$= \frac{1}{20}\ln\left(8x^2+x+1\right) - \frac{3}{20}\frac{2}{\sqrt{31}}\left(arctg\frac{16x+1}{\sqrt{31}} - arctg\frac{1}{\sqrt{31}}\right)$$

Substitute (9), (10) to (2), we have:

$$\varphi = \frac{J_0\omega_0^2}{10M_0}\left[\frac{1}{2}\ln\left(8x^2+x+1\right) - \ln\left|x-1\right| - \frac{3}{\sqrt{31}}\left(arctg\frac{16x+1}{\sqrt{31}} - arctg\frac{1}{\sqrt{31}}\right)\right] \tag{11}$$

Function (11) is for the rotary angle $\varphi = \varphi(x)$ so the inverse function is $x = x(\varphi)$ or $\omega = \omega(\varphi)$