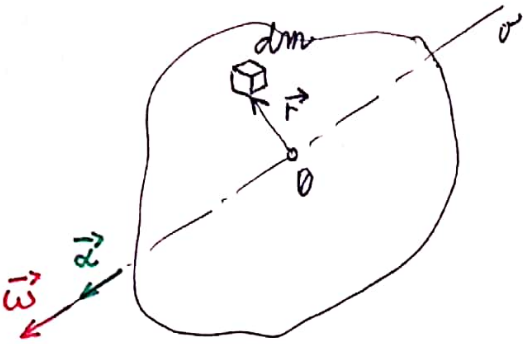
# ROTATING BODY - LAW OF CHANGE OF ANGULAR MOMENTUM



$$\frac{d\vec{L}}{dt} = \vec{M}^E$$




$\vec{M}^E$ ... sum of moments of external action and reaction forces acting to body.

$\vec{L}$ ... angular momentum

$$d\vec{L} = \vec{r}\times d\vec{p} = \vec{r}\times(dm\cdot\vec{v}) =$$
$$= \vec{r}\times(\vec{\omega}\times\vec{r})\cdot dm$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\int_{(m)} \vec{r}\times(\vec{\omega}\times\vec{r})\, dm =$$

$$= \int_{(m)} \Big[ \underbrace{\vec{v}\times\underbrace{(\vec{\omega}\times\vec{r})}_{\vec{v}}}_{\vec{0}} + \vec{r}\times(\vec{\alpha}\times\vec{r}) + \vec{r}\times(\vec{\omega}\times\vec{v}) \Big]\, dm =$$

$$\boxed{\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}}$$

$$= \int_{(m)} \Big[ r^2\vec{\alpha} - (\vec{r}\cdot\vec{\alpha})\vec{r} + \vec{r}\times\{ \vec{\omega}\times(\vec{\omega}\times\vec{r}) \} \Big]\, dm =$$

$$= \int_{(m)} \Big[ r^2\vec{\alpha} - (\vec{r}\cdot\vec{\alpha})\vec{r} + \vec{r}\times\{ (\vec{\omega}\cdot\vec{r})\vec{\omega} - \omega^2\vec{r} \} \Big]\, dm =$$

$$= \int_{(m)} \Big[ r^2\vec{\alpha} - (\vec{r}\vec{\alpha})\vec{r} + (\vec{r}\vec{\omega})\cdot(\vec{r}\times\vec{\omega}) - \underbrace{\vec{r}\times(\omega^2\vec{r})}_{\vec{0}} \Big]\, dm = *$$

for simplification:

$o \equiv x^G \equiv x$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{\omega} = \omega\vec{i}$

$\vec{\alpha} = \alpha\vec{i}$

$$\vec{r}\cdot\vec{r} = (x, y, z)\cdot(x, y, z) = x^2 + y^2 + z^2 = r^2$$

$$\vec{r}\cdot\vec{\alpha} = (x\vec{i} + y\vec{j} + z\vec{k})\cdot\alpha\vec{i} = x\cdot\alpha \qquad (\vec{i}\cdot\vec{i} = 1)$$

$$\vec{r}\cdot\vec{\omega} = (x\vec{i} + y\vec{j} + z\vec{k})\cdot\omega\vec{i} = x\cdot\omega$$

$$\vec{r}\times\vec{\omega} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ \omega & 0 & 0 \end{vmatrix} = \vec{i}(y\cdot 0 - z\cdot 0) - \vec{j}(x\cdot 0 - z\omega) +$$
$$+ \vec{k}(x\cdot 0 - y\cdot\omega) =$$
$$= \vec{j}\omega z - \vec{k}\omega y$$

$$* = \int_{(m)} (x^2+y^2+z^2)\,\alpha\,\vec{i} - x\cdot\alpha\,(x\vec{i}+y\vec{j}+z\vec{k}) + x\omega\cdot(z\omega\,\vec{j} - y\omega\,\vec{k})\,]\,dm =$$

$$= \left(\alpha \int_{(m)} (y^2+z^2)\,dm\right)\vec{i} - \left(\alpha\int_{(m)} xy\,dm - \omega^2\int_{(m)} xz\,dm\right)\vec{j} +$$

$$+ \left(\alpha\int_{(m)} xz\,dm + \omega^2\int_{(m)} xy\,dm\right)\vec{k} =$$

$$= J_x\,\alpha\,\vec{i} - (\alpha D_{xy} - \omega^2 D_{xz})\,\vec{j} - (\alpha D_{xz} + \omega^2 D_{xy})$$

we use this result : $\dfrac{d\vec{L}}{dt} = \vec{M}^E$

$$J_x\alpha\,\vec{i} - (\alpha D_{xy} - \omega^2 D_{xz})\,\vec{j} - (\alpha D_{xz} + \omega^2 D_{xy}) = \vec{M}^E$$

$$\overbrace{-J_x\,\alpha\,\vec{i} + (\alpha D_{xy} - \omega^2 D_{xz})\,\vec{j} + (\alpha D_{xz} + \omega^2 D_{xy})}^{\vec{M}_D} + \vec{M}^E = \emptyset$$

Eq. of Equlibrium of moments :

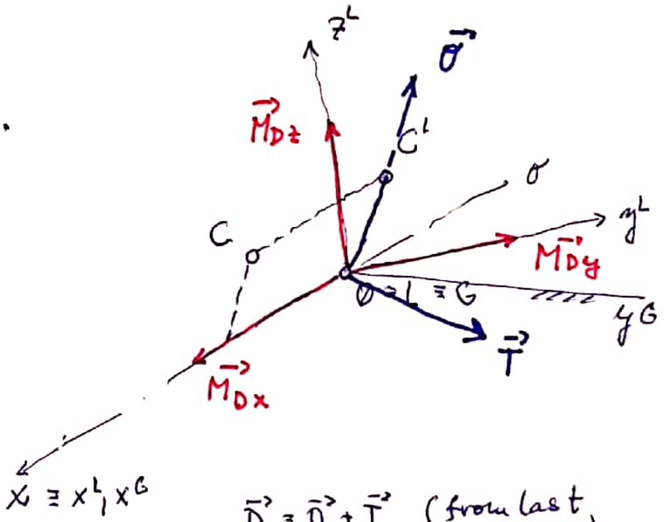
$\vec{M}_D$ ... d'Alemberts moment.

$$\vec{M}_D = M_{Dx}\,\vec{i} + M_{Dy}\,\vec{j} + M_{Dz}\,\vec{k}$$

$$M_{Dx} = -J_x\,\alpha$$

$$M_{Dy} = \alpha D_{xy} - \omega^2 D_{xz}$$

$$M_{Dz} = \alpha D_{xz} + \omega^2 D_{xy}$$




$\vec{D} = \vec{O} + \vec{T}$ (from last lesson)

$(L, x^L, y^L, z^L)$ ... coordinate (local) system is connected with rotating body

C ... center of mass

C' ... projection of C to plane $(z^L, y^L)$

$(G, x^G, y^G, z^G)$ ... coordinate (global) system is connected with ground