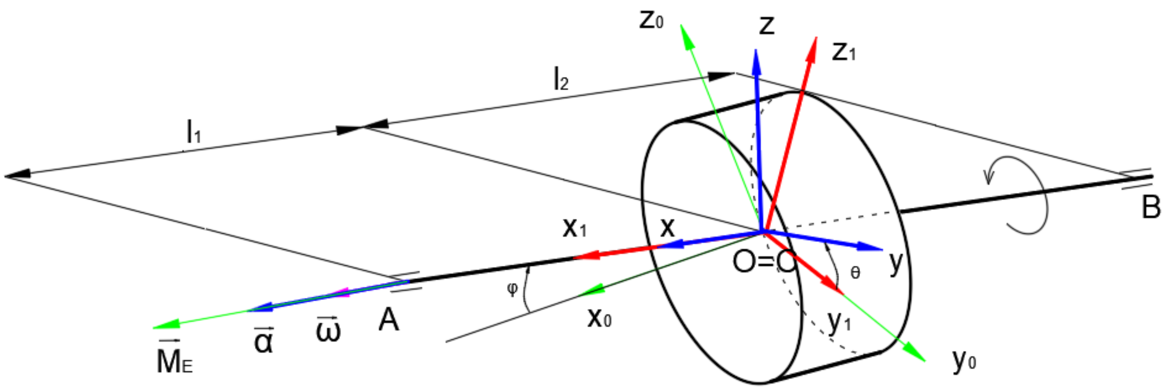
Dynamics of rotational motion of a body

The skew cylinder is attached to the vertical rotary axis as in the figure.

Given:

- Dimension of the cylinder *h*, *r*;
- Mass of the cylinder m;
- Skew angle φ
- Dimension of the axis l_1, l_2;
- Driving moment to the rotary axis M_E



Task:

- Moment of inertia about the axis
- System of equations of motion

Solution:

The local coordinate system O, x_0, y_0, z_0 is attached to the cylinder.

We know the matrix of inertia of the cylinder to the coordinate system O, x_0, y_0, z_0 as following:

$$I_0 = \frac{1}{4} m \begin{bmatrix} 2r^2 & 0 & 0 \\ 0 & r^2 + \frac{h^2}{3} & 0 \\ 0 & 0 & r^2 + \frac{h^2}{3} \end{bmatrix} \quad (1)$$

The second coordinate system O, x_1, y_1, z_1 which has $y_1 \equiv y_0$ is attached to the rotary axis.

Matrix of transformation from the coordinate system O, x_0, y_0, z_0 to the coordinate system O, x_1, y_1, z_1 :

$$T_1 = \begin{bmatrix} \cos(-\varphi) & 0 & \sin(-\varphi) \\ 0 & 1 & 0 \\ -\sin(-\varphi) & 0 & \cos(-\varphi) \end{bmatrix} = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \tag{2}$$

Then we find the matrix of inertia of the cylinder in the coordinate system O, x_1, y_1, z_1 is:

$$I_1 = T_1^T I_0 T_1 \tag{3}$$

$$I_1 = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \frac{1}{4} m \begin{bmatrix} 2r^2 & 0 & 0 \\ 0 & r^2 + \frac{h^2}{3} & 0 \\ 0 & 0 & r^2 + \frac{h^2}{3} \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \tag{4}$$

$$= \frac{1}{4} m \begin{bmatrix} \left(r^2 + \frac{h^2}{3}\right)\sin^2\varphi + 2r^2\cos^2\varphi & 0 & \left(-r^2 + \frac{h^2}{3}\right)\sin\varphi\cos\varphi \\ 0 & r^2 + \frac{h^2}{3} & 0 \\ \left(-r^2 + \frac{h^2}{3}\right)\sin\varphi\cos\varphi & 0 & \left(r^2 + \frac{h^2}{3}\right)\cos^2\varphi + 2r^2\sin^2\varphi \end{bmatrix}$$

The global coordinate system O, x, y, z which has $x \equiv x_1$ is fixed on the space.

The rotary axis rotates about the x-axis with angle θ.

Matrix of transformation from the coordinate system O, x_1, y_1, z_1 to the coordinate system O, x, y, z :

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{5}$$

Then we find the matrix of inertia of the cylinder in the coordinate system O, x, y, z is:

$$I = T_2^T I_1 T_2 \tag{6}$$

So we have moment of inertia with respect to the x-axis as follow:

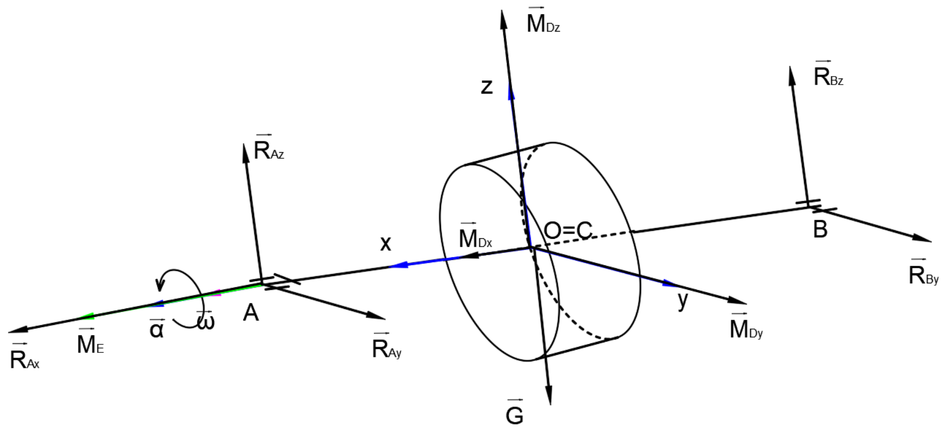
$$J_x = \frac{m}{4}\left[\left(r^2 + \frac{h^2}{3}\right)\sin^2\varphi + 2r^2\cos^2\varphi\right]; \qquad (7)$$

And

$$D_{xy} = \frac{m}{4}\sin\theta\cos\varphi\sin\varphi\left(-r^2 + \frac{h^2}{3}\right)$$
$$D_{xz} = \frac{m}{4}\cos\theta\cos\varphi\sin\varphi\left(-r^2 + \frac{h^2}{3}\right)$$

Dynamics of rotational motion of the cylinder about the rotary axis:

The free body diagram:



Equations are in vector form:

$$\sum \vec{F}_i^E + \vec{D} = \vec{0} \qquad (8)$$
$$\sum \vec{M}_i^E + \vec{M}_D = \vec{0} \qquad (9)$$

where:

$\vec{F}_i^E$ and $\vec{M}_i^E$ are external force and external moment,

$\vec{D}$ and $\vec{M}_D$ are D’Alembert force and D’Alembert moment

For rotating body, we know:

$$\vec{D} = \vec{T} + \vec{O} \qquad (10)$$
$$\vec{T} = -m\vec{a}_{ct} = -m\left(\vec{\alpha} \times \vec{r}_C\right) \qquad (11)$$
$$\vec{O} = -m\vec{a}_{Cn} = -m\left(\vec{\omega} \times \vec{v}_C\right) \qquad (12)$$

Where:

$\vec{T}$ and $\vec{O}$ are vectors of tangent component and normal component of D'Alembert force,

$\vec{a}_{Ct}$ and $\vec{a}_{Cn}$ are vectors of tangent acceleration and normal acceleration in the coordinate system O, x_1, y_1, z_1 ,

$\vec{\alpha}$ and $\vec{\omega}$ are vectors of angular acceleration and angular velocity of the cylinder in the coordinate system O, x_1, y_1, z_1 ,

$\vec{r}_C$ and $\vec{v}_C$ are vectors of displacement and velocity of the center point of mass of the cylinder in the coordinate system O, x_1, y_1, z_1 .

Because of $C \equiv O$, we have:

$$\vec{r}_C = \vec{0} \quad (13)$$
$$\vec{v}_C = \vec{0}$$

So we get:

$$\vec{D} = \vec{T} = \vec{O} = \vec{0} \quad (14)$$

The system of equations is given from the component equations of (6) and (7) as follows:

$$(x_1): \mathrm{R}_{Ax} = 0 \quad (15)$$
$$(y_1): R_{Ay} + R_{By} = 0 \quad (16)$$
$$(z_1): R_{Az} + R_{Bz} - G = 0 \quad (17)$$
$$\left(\widehat{M}_{x_1}\right): M_E + M_{Dx} = 0 \quad (18)$$
$$\left(\widehat{M}_{y_1}\right): \mathrm{R}_{Bz} l_2 - \mathrm{R}_{Az} l_1 + M_{Dy} = 0 \quad (19)$$
$$\left(\widehat{M}_{z_1}\right): \mathrm{R}_{Ay} l_1 - \mathrm{R}_{By} l_2 + M_{Dz} = 0 \quad (20)$$

Where:

$$G = mg \quad (21)$$
$$M_{Dx} = -J_x \alpha; \quad (22)$$
$$M_{Dy} = D_{xy}\alpha - D_{xz}\omega^2 = \frac{m}{4}\cos\varphi\sin\varphi\left(-r^2 + \frac{h^2}{3}\right)\left(\alpha\sin\theta - \omega^2\cos\theta\right); \quad (23)$$
$$M_{Dz} = D_{xz}\alpha + D_{xy}\omega^2 = \frac{m}{4}\cos\varphi\sin\varphi\left(-r^2 + \frac{h^2}{3}\right)\left(\omega^2\sin\theta + \alpha\cos\theta\right). \quad (24)$$
$$\alpha = \dot{\omega} \quad (25)$$

In which J_x, D_{xy}, D_{xz} given from (5).

Solving the system of equations (16), (17), (18), (19), (20) with considering to (7), (21), (22), (23), (24) ,(25), we get:

$$\alpha = \frac{M_E}{J_x};$$

$$\alpha = \dot{\omega} = \omega \frac{d\omega}{d\theta} \Rightarrow \alpha d\theta = \omega d\omega \Rightarrow 2\alpha d\theta = d\omega^2 \Rightarrow \omega^2 = 2\alpha\theta$$

$$\mathrm{R}_{Ay} = -\frac{D_{xz}\alpha + D_{xy}\omega^2}{(l_1 + l_2)}, \ \mathrm{R}_{Az} = \frac{-D_{xy}\alpha + D_{xz}\omega^2 + mgl_1}{l_1 + l_2}$$

$$\mathrm{R}_{By} = \frac{D_{xz}\alpha + D_{xy}\omega^2}{(l_1 + l_2)}, \ \mathrm{R}_{Bz} = \frac{D_{xy}\alpha - D_{xz}\omega^2 + mgl_2}{l_1 + l_2}$$

Where θ is the rotated angle of the rotary axis ($0 \le \theta \le 2\pi$)