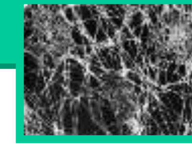

Ohebnost (flexibilita) vlákenných útvarů

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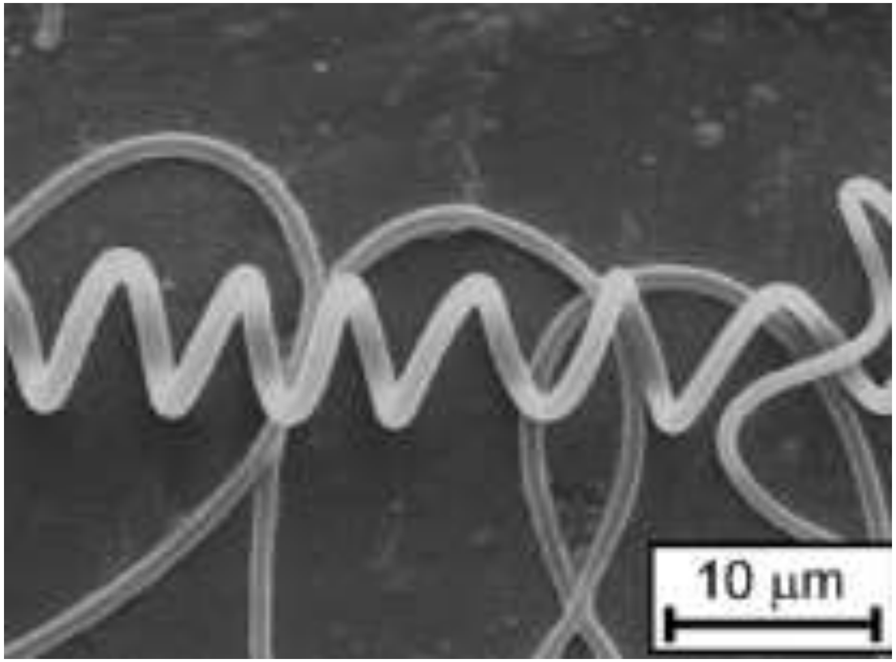
KCH, FP, TUL

Na základě přednášky: FIBER FLEXIBILITY
TEXT 822, Lect7, Sirrine 260
Instructor: Dr. Kostya Kornev
Clemson University

Advantages of fibers – výhody vláken

One of the most important advantages of fibers is their flexibility, i.e. their ability to bent and form loops and knots with the radius compared to their diameter.
This astonishing property of many materials which form fiber-like or beam shape, has drew attention of many great scientists: Galileo Galilei, Leonardo da Vinci, and Hooke.



History

Galileo Galilei made the first attempts at developing a theory of beams, but Galileo was held back by an incorrect assumption he made.

Leonardo da Vinci was the first to make the crucial observations. Da Vinci lacked Hooke's law and calculus to complete the theory.

Bernoulli-Euler beam theory

Euler–Bernoulli beam theory is a simplification of the linear theory of elasticity which provides a means of calculating the load-carrying and deflection characteristics of beams.

It covers the case for small deflections of a beam which is subjected to lateral loads only.

It was first enunciated circa 1750, but was not applied on a large scale until the development of the Eiffel Tower and the Ferris wheel in the late 19th century. Following these successful demonstrations, it quickly became a cornerstone of engineering and an enabler of the Second Industrial Revolution.

A Ferris wheel is a nonbuilding structure consisting of a rotating upright wheel with passenger cars attached to the rim. The original Ferris Wheel was designed and constructed by George Washington Gale Ferris, Jr. as a landmark for the 1893 World's Columbian Exposition in Chicago.

Leonhard Euler and Daniel Bernoulli theory

Leonhard Euler and Daniel Bernoulli were the first to put together a useful theory circa 1750.

At the time, science and engineering were generally seen as very distinct fields, and there was considerable doubt that a mathematical product of academia could be trusted for practical safety applications.

Bridges and buildings continued to be designed by precedent until the late 19th century, when the **Eiffel Tower and Ferris wheel demonstrated the validity of the theory on large scales.**

Gustave Eiffel

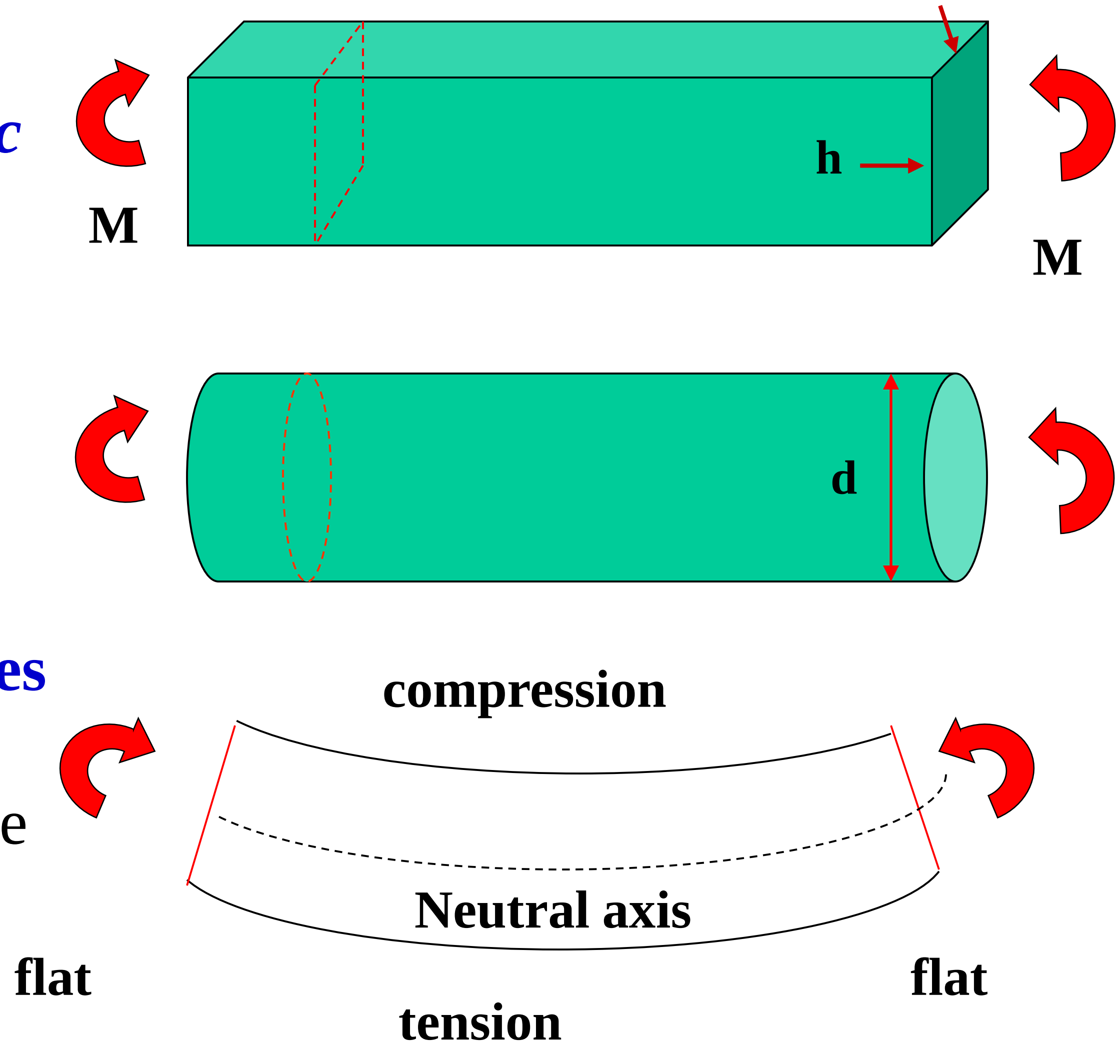
Gustave Eiffel, Maria Pia Bridge, Porto, Portugal, [1877]

http://www.columbia.edu/cu/gsapp/BT/STRUCTI/WEEK1/structures1.html

Two basic assumptions underlie the beam theory

1. The **material is *isotropic* and obeys the linear Hooke’s law** of elasticity. These properties remain unchanged during the deformations;

2. The **flat** transverse **planes** of element in the unloaded beam **will still be flat** in the bent beam.



Two basic assumptions underlie the beam theory

These assumptions are adequate for metals and glasses but hardly applicable for many high performance polymer and carbon fibers, even if the fibers deform linearly.

Internal structure of these fibers is highly anisotropic and still poorly understood.

However, the beam theory is proven suitable for description of many, even anisotropic, fibers.

A fiber bent in a radius R

Fig.1: The loop of a bended fiber lies in the xy-plane.

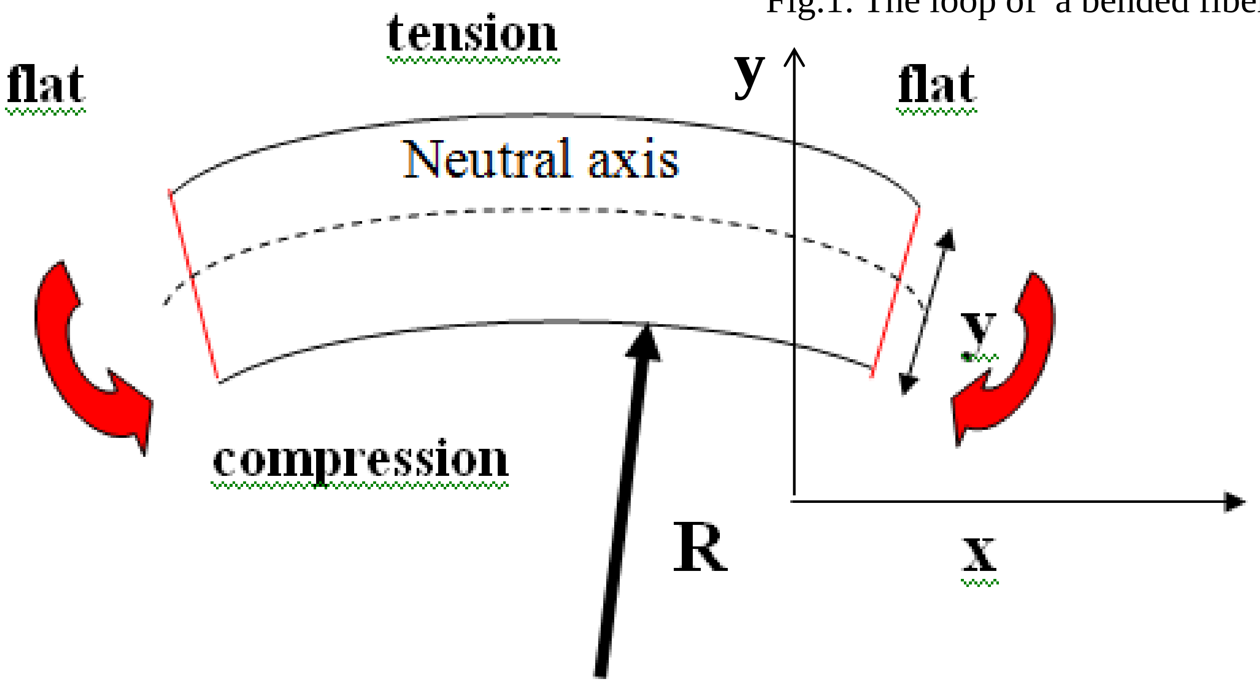
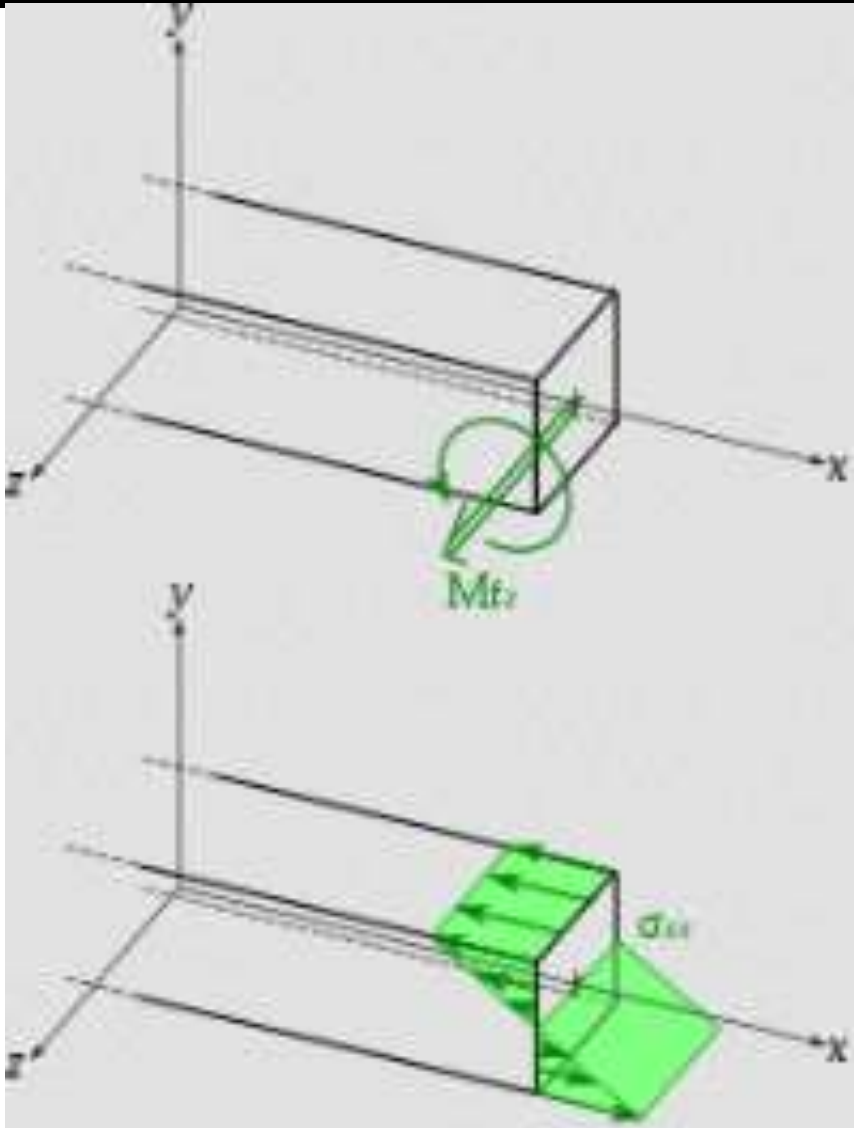
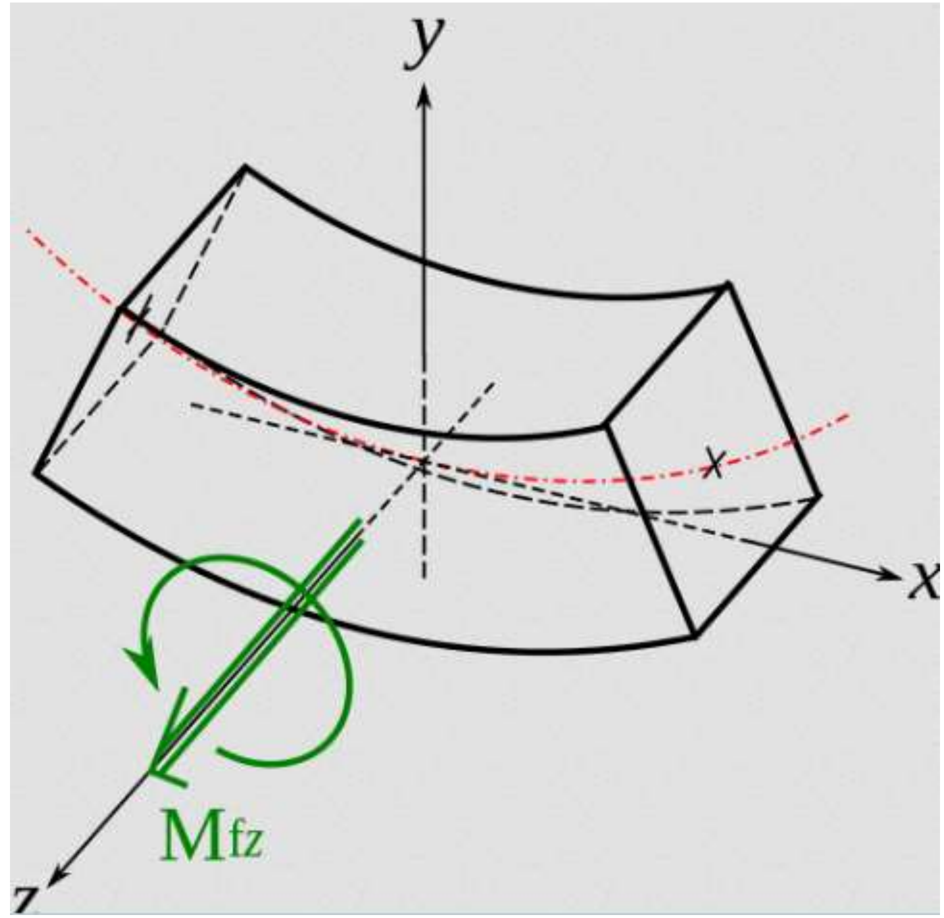


Fig.1a: Schematic of curved fiber, R>> fiber diameter.

We assume that bending moments M are applied to the fiber ends and we would like to find a relation between the bending moments M and the loop radius R.

http://en.wikipedia.org/wiki/Bending



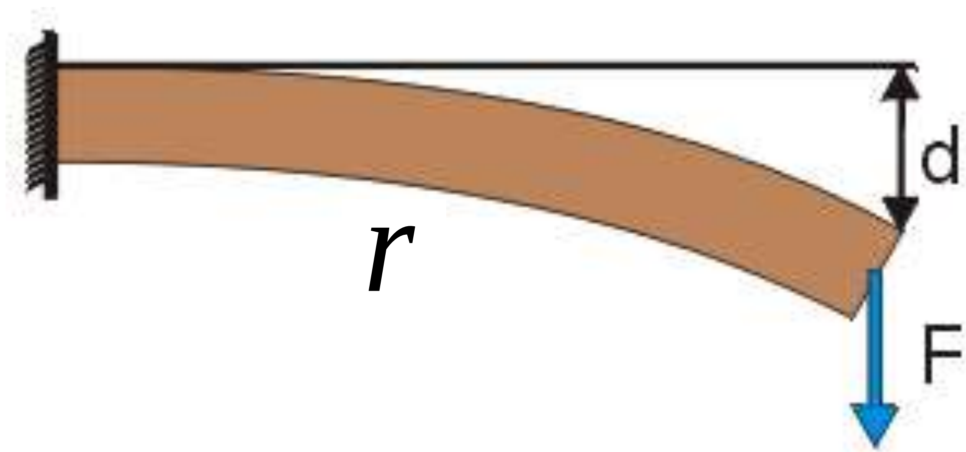


Conditions of static equilibrium

Ohybový moment je statická veličina. Jde o moment síly způsobující ohyb prvku (trámu, desky apod). Značí se M a základní jednotka je Newton na metr.

Tato veličina se používá k dimenzování nosných konstrukcí jak ve stavebnictví, tak i ve strojírenství.

$$\vec{M} = \vec{r} \times \vec{F}$$



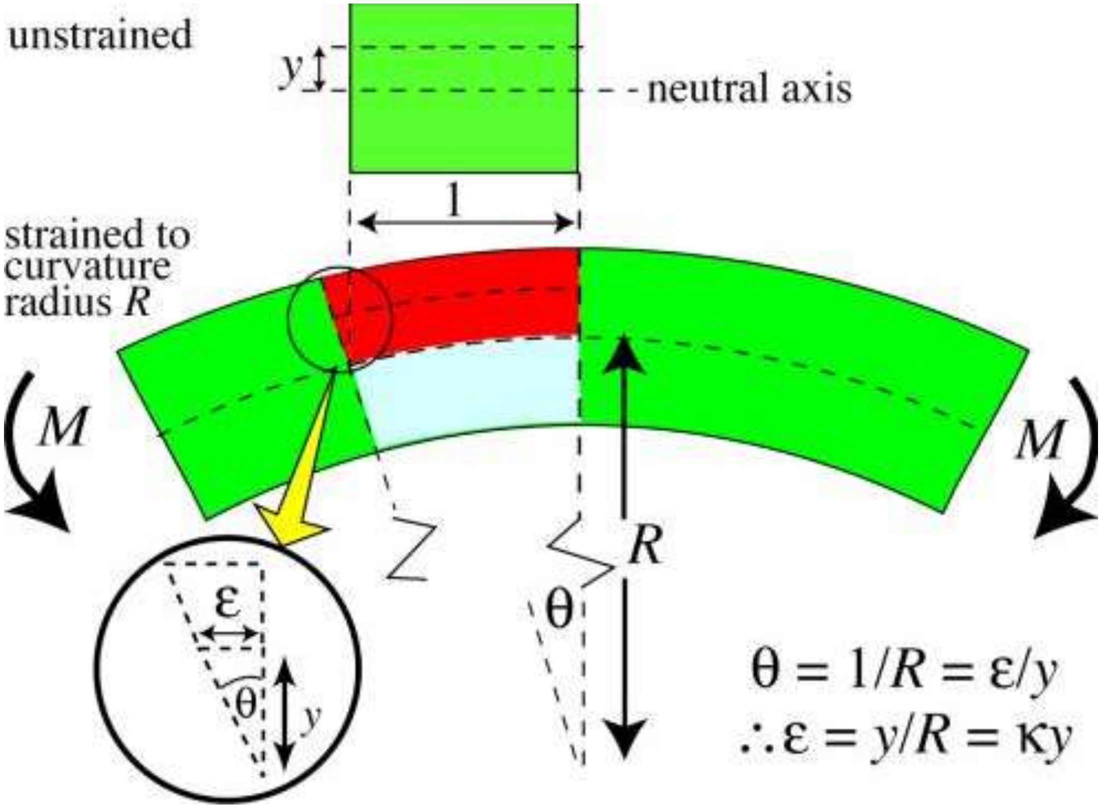
Mechanická rovnováha :

Stav, kdy výslednice sil a momentů sil působící na fyzikální soustavu, například na hmotné těleso, je nulová.

$$\sum \vec{M} = 0$$

Stretched and compressed parts of a beam

Bending has stretched the upper parts of the beam and compressed the lower parts.



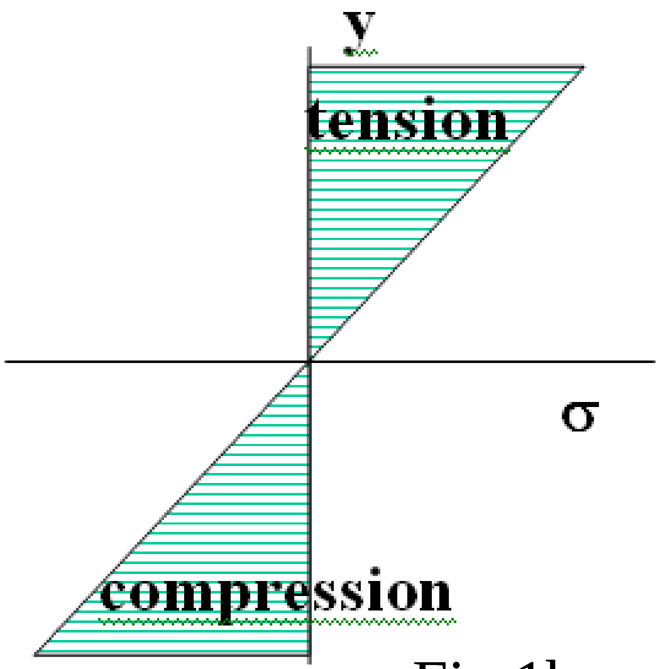
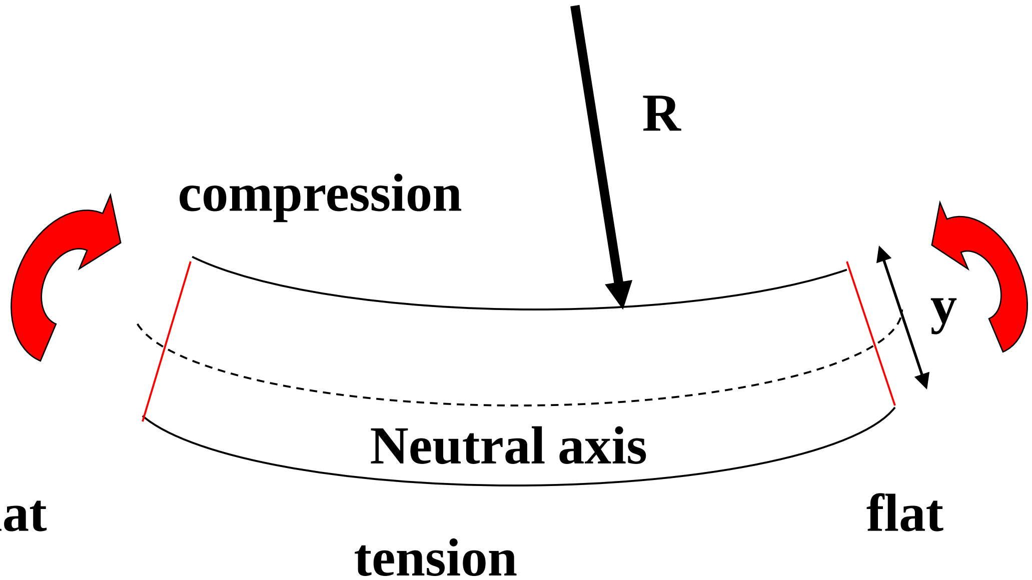
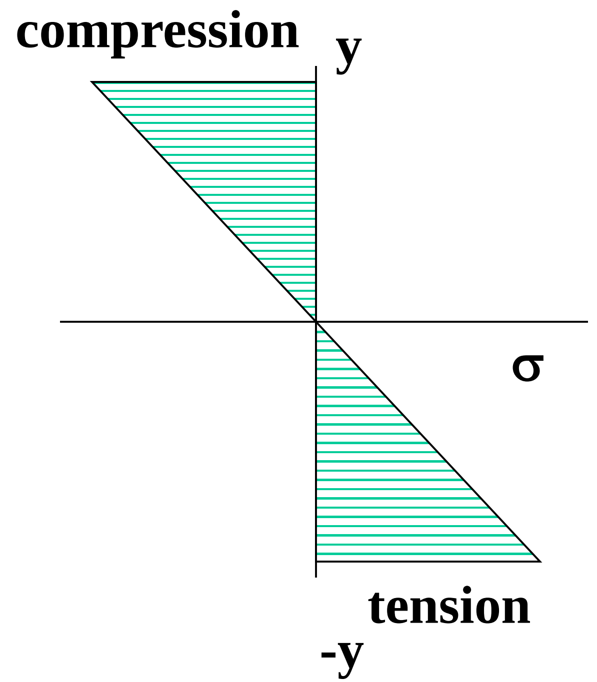


Fig.1b

Since the strain is proportional to the tensile stress, the distribution of tensile stress across the fiber must be as shown in Fig.1b) .

Neutral axis does not change its length



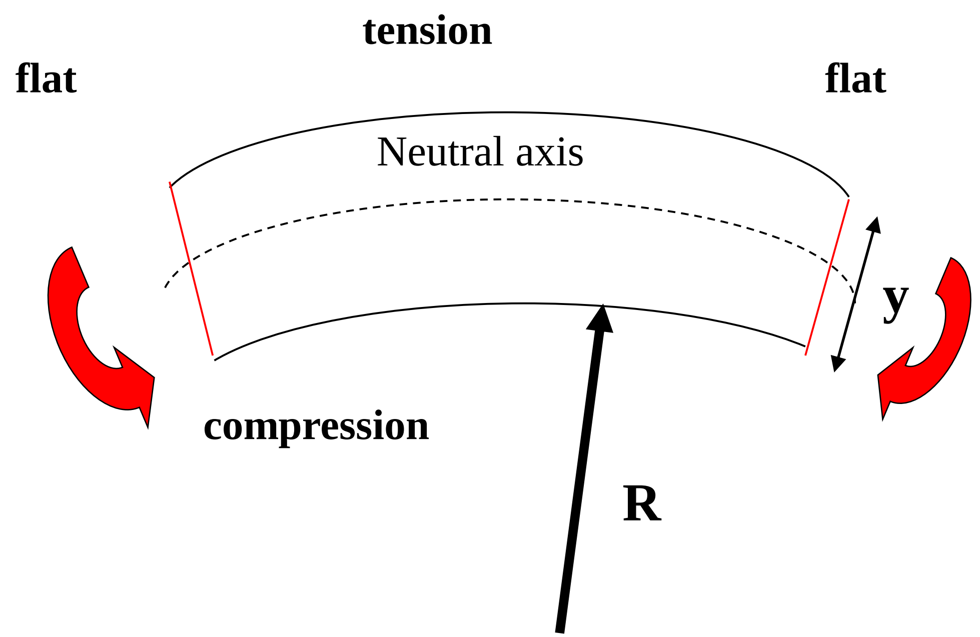


Stress distribution about the neutral axis

There is **an intermediate filament** in the fiber (or a layer if the fiber has rectangular cross-section) **which is neither extended or compressed**. It is called **neutral axis** (or surface).

Stresses in a loop of radius R

Neutral axis does not change its length: 2πR

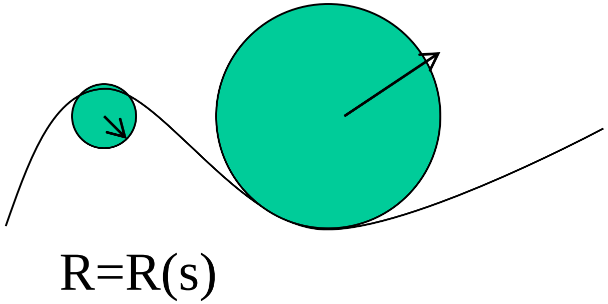


Other filaments change its length: **(2πR ± 2πy)- 2πR= ± 2πy**,

and suffer the **strain**: $$\varepsilon_{zz} = \frac{l - l_0}{l_0} = \frac{2\pi(R \pm y) - 2\pi R}{2\pi R} = \pm\frac{y}{R}$$

and produce the stress: $$\sigma_{zz} = E\varepsilon_{zz} = \pm E\frac{y}{R} \Rightarrow \boxed{\pm\frac{\sigma_{zz}}{y} = \frac{E}{R}} \quad (7.2)$$

More complicated fiber shape



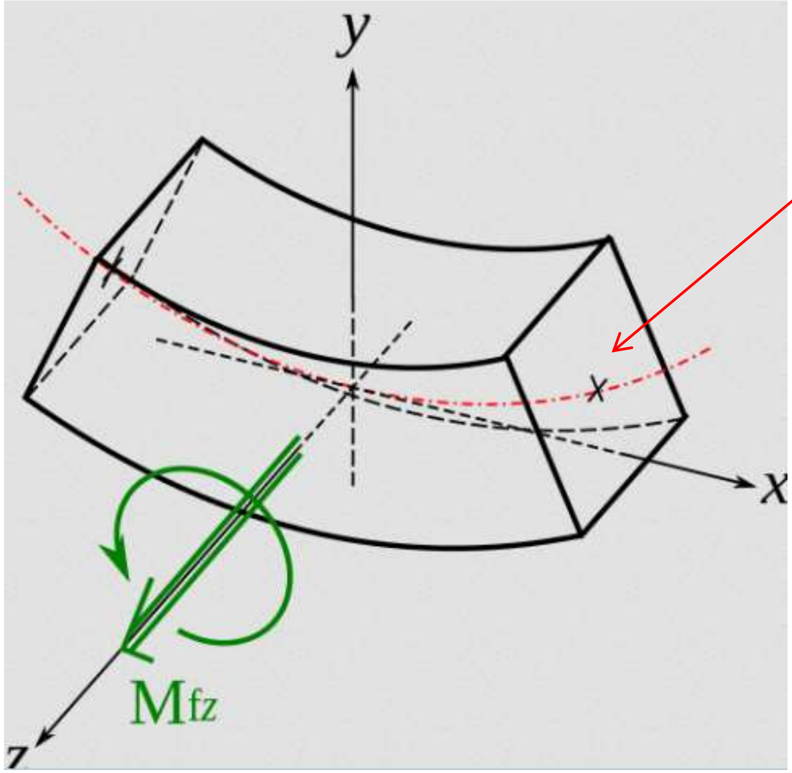
$$\pm \frac{\sigma_{zz}}{y} = \frac{E}{R} \qquad (7.2)$$

This relation holds true for any point along the fiber, even if it is **bent in a more complicated shape**, provided that R=R(s), i.e. the local radius depends on the arc length s.

Total force acting on this cross-section should be zero

Due to the second assumption, if the transverse planes are to remain plane, the total force acting on this cross-section should be zero.

2. The **flat** transverse **planes** of element in the unloaded beam **will still be flat** in the bent beam.



First moment of inertia

Therefore, the force distribution shown in Fig. 1 should give no resultant force. Since the tensile force acting on the infinitesimal slice with the cross sectional area dA is written as

$$dF = \sigma_{zz}\, dA, \qquad (7.3)$$

the total force is zero

$$\int dF = \int \sigma_{zz}\, dA = 0\ ,$$

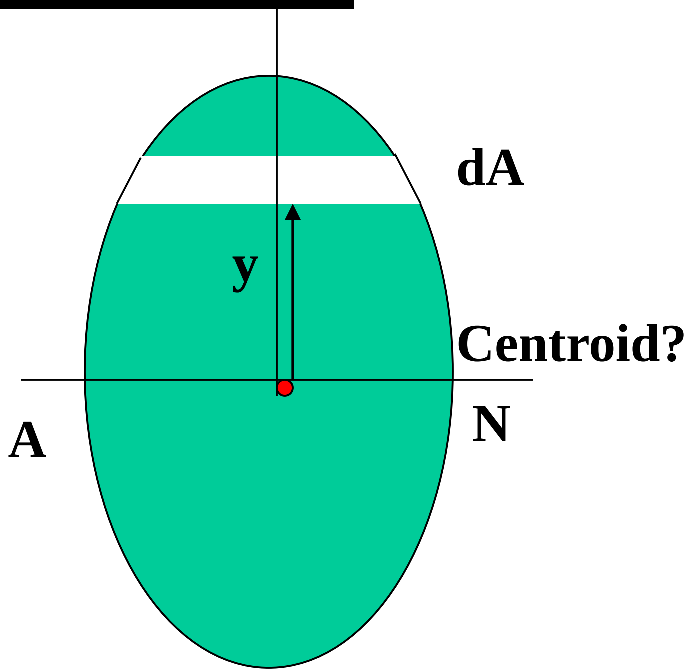
or, using (7.2) , we have

$$\pm\frac{\sigma_{zz}}{y} = \frac{E}{R} \Rightarrow \sigma_{zz} = \frac{Ey}{R} \qquad (7.2)$$

$$\int_A \sigma_{zz}\, dA = E \int_A \frac{y}{R}\, dA = 0 \qquad (7.4)$$

This equation serves for specification of the neutral axis (surface) and the quantity **$I_y = \int y\,dA$** is called the **first moment of inertia**.

Neutral axis



Take a strip of area dA positioned at the distance y from AN and || to it. The total force in the strip is

$$dF = \sigma_{zz} dA$$

The total force F acting on the cross section must be zero

$$F = \int_A \sigma_{zz} dA = 0 \Rightarrow \int y \, dA = 0$$

This is the first moment I_y of inertia (with respect to x-axis). This equation is identical to the condition that the neutral axis passes through the center of gravity of this cross-section.

$$I_y = \int y dA$$

$$\sigma_{zz} = \frac{Ey}{R}$$

Bending moment

$$\pm \frac{\sigma_{zz}}{y} = \frac{E}{R} \quad (7.2)$$

We are in position to find the **bending moment M as a function of the loop radius R**. Consider the force moment as a resultant of the stress distribution (7.2). In any fiber cross-section perpendicular to the loop plane and neutral axis, the force moment is written as

$$dM = y\sigma_{zz} dA. \quad (7.6)$$

Substituting (7.2), we have $dM = Ey^2 dA/R$. Integrating this relation, we come to the **Bernoulli-Euler** result:

$$M = \int_A dM = \int_A \frac{Ey}{R} y dA = \frac{E}{R} \int_A y^2 dA = \frac{EI}{R} \quad (7.7)$$

$$M = \frac{EI}{R}$$

where I is the second moment of inertia

$$I = \int y^2 dA. \quad (7.8)$$

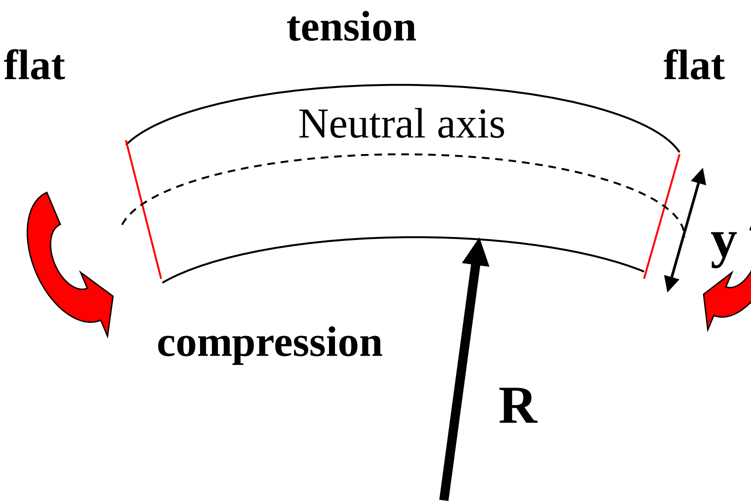
As you see, the bending moment has the right dimension

$[M] = [Nm^{-2}]\ [m^4]\ /\ [m] = [N{*}m]$

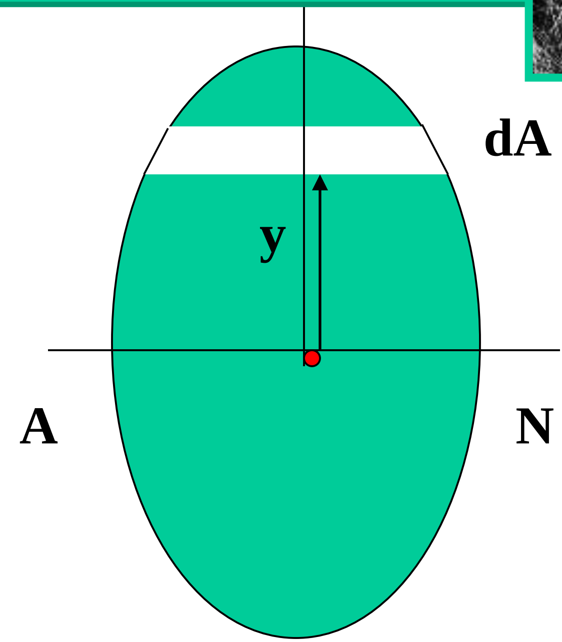
$$I_y = \int y dA$$

The product EI is called **flexural rigidity** and shows the fiber ability to resist bending.

Bending moment: M=EI/R



flat



Force in elementary strip: $dF = \sigma_{zz} dA$

Moment of force about AN: $dM = y\sigma_{zz} dA$

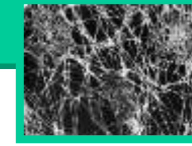
$I = \int y^2 dA$

Second moment of inertia

Since $\sigma_{zz} = y\sigma_{tens}/y_{tens}$

Moment on strip: $dM = (\sigma_{comp}/y_{comp})y^2 dA = (\sigma_{tens}/y_{tens})y^2 dA$

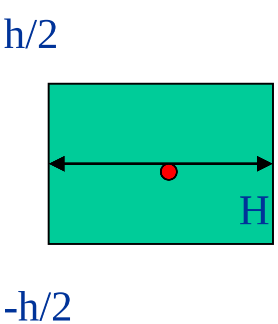
$$M = (\sigma_{comp}/y_{comp}) \int y^2 dA = (\sigma_{tens}/y_{tens}) \int y^2 dA = EI/R$$



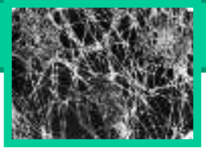
Example 2:

We have a fiber with a **rectangular cross-section**, *h* x *H*, and place the origin of coordinates at its center (x = 0, y = 0).

$$I = \int_A y^2 dA = \int_{-H/2}^{H/2} dx \int_{-h/2}^{h/2} y^2 dy = \left. \frac{Hy^3}{3} \right|_{-h/2}^{h/2} = \frac{Hh^3}{24} + \frac{Hh^3}{24} = \frac{Hh^3}{12} \qquad (7.9)$$



$$I = \frac{Hh^3}{12}$$



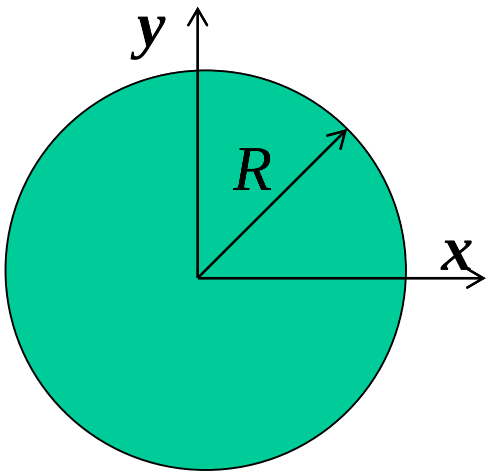
Example 3:

We have a fiber with a **circular cross-section** with the radius R and place the origin of coordinates at its center (x = 0, y = 0).

$$I = \int_{-H/2}^{H/2} dx \int_{-h/2}^{h/2} y^2 dy$$

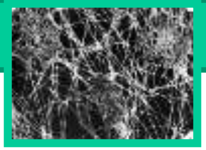
$$I = \frac{R^4}{4}\pi$$

$$y = r\sin\theta, x = r\cos\theta, dA = rdrd\theta$$

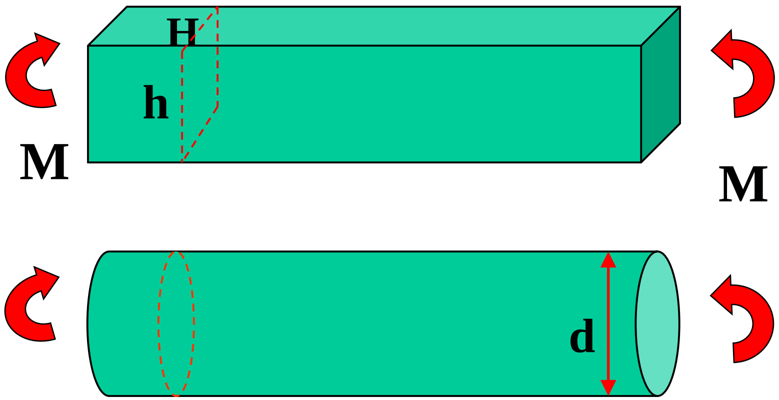


$$I = \int_0^R \int_0^{2\pi} r^2 \sin^2\theta \, rdrd\theta = \qquad (7.10)$$

$$= \int_0^R r^3 dr \int_0^{2\pi} \sin^2\theta \, d\theta = \frac{R^4}{4}\pi$$



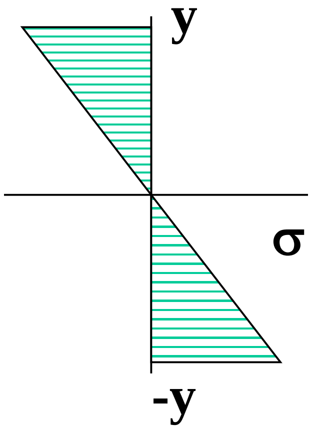
Summary of the Euler-Bernulli theory



Second moment of inertia

$$I = \frac{Hh^3}{12}$$

$$I = \frac{\pi R^4}{4} = \frac{\pi d^4}{64}$$



axial stress

$$\sigma_{zz} = My/I = E\,y/R$$

Bending moment

$$M = \frac{EI}{R}$$

Units:

$$[M] = \frac{\frac{N}{m^2} m^4}{m} = N \cdot m$$

Flexibility

E = Young’s modulus, material parameter

EI = flexural rigidity (≈ resistance of beam to bending),

i.e., material parameter x shape factor

1/R = curvature, visible reaction

Flexibility f– the ability of a fiber to be bent, i.e. the inverse of the product,

$$f = \frac{1}{MR} = \frac{R}{EIR} = \frac{1}{EI}$$

would be the criterion of fiber flexibility.

$M \to 0$, $R \to 0$ (we can make loops, knots, can weave, coil,…)
$\Rightarrow f \to \infty$

Fiber flexibility

As a measure of fiber flexibility it is convenient to choose f

$$f = \frac{1}{MR} = \frac{1}{EI}$$

In this definition, the smaller the fiber reaction on applied bending moment the greater the flexibility.

Also, the smaller the loop which this fiber can form, the greater its flexibility.

Hence, this complex f reflects our understanding of flexibility.

great flexibility of nanometer thick fibers

The flexibility of cylindrical fibers is inversely proportional to the Young's modulus of the material and the fiber diameter to the fourth.

$$I = \frac{\pi R^4}{4} = \frac{\pi d^4}{64}$$

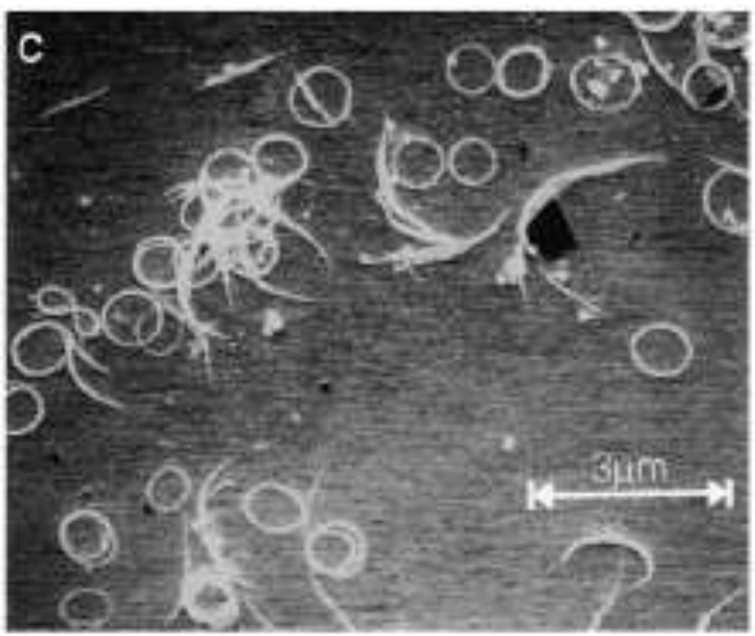
It is expected therefore, that the nanometer thick fibers will have the greatest flexibility possible among other fibers.

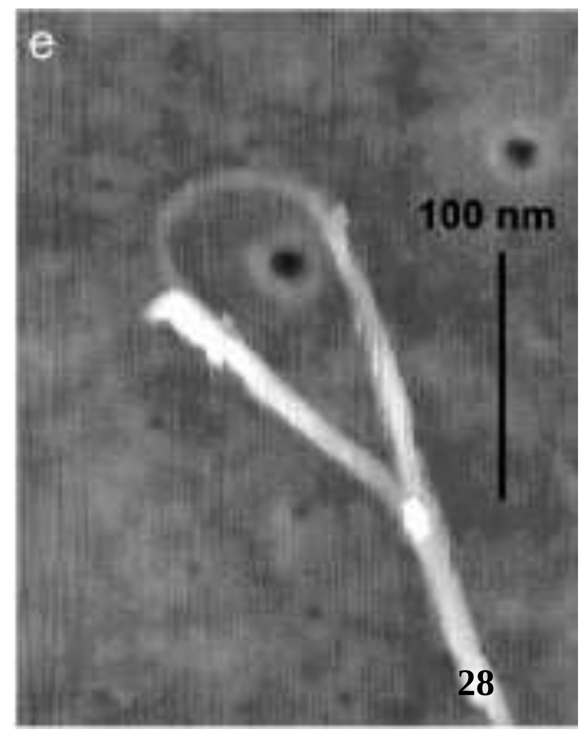
spontaneously coiled nanotubes

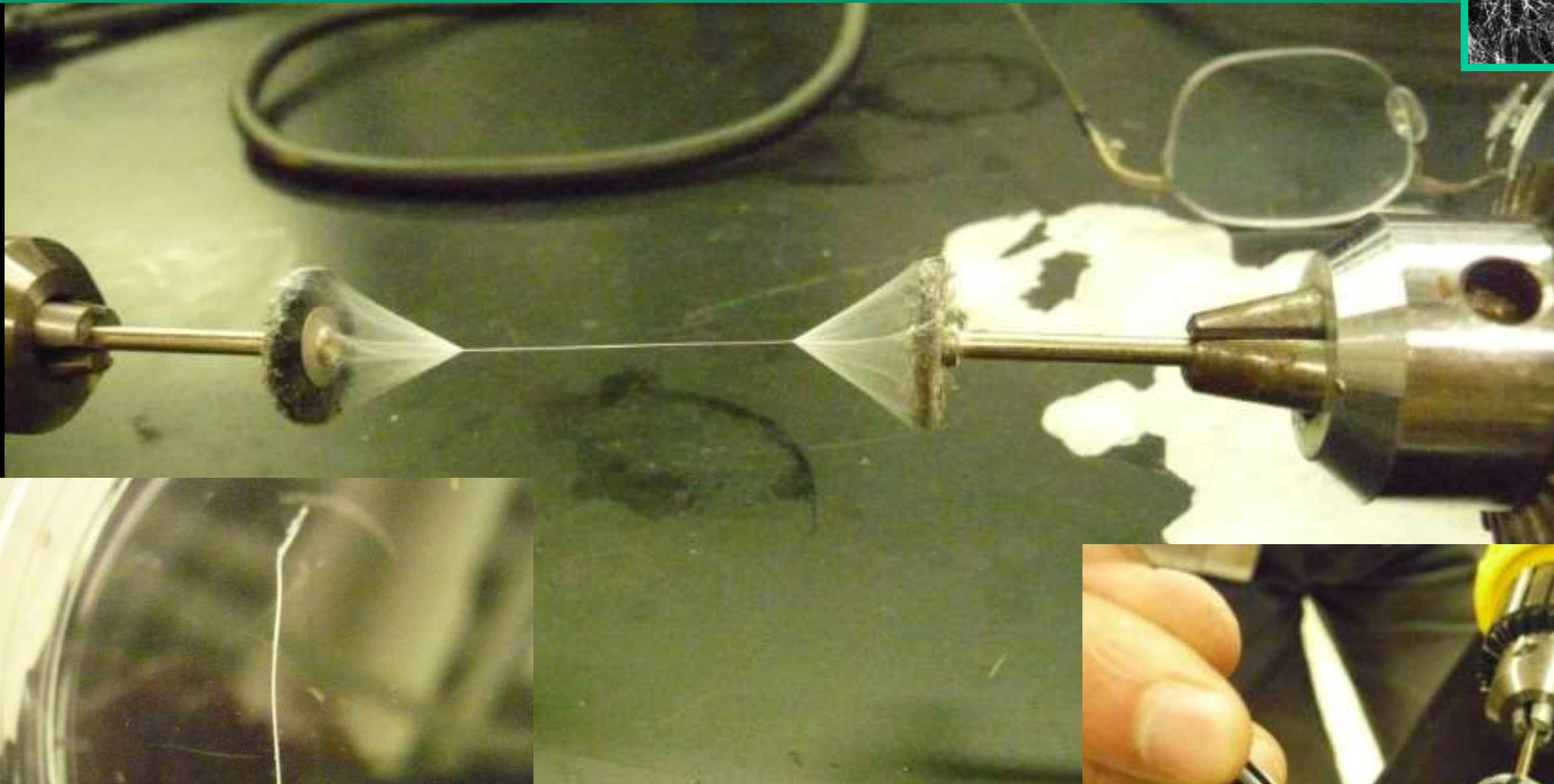
The pictures taken from Ref. (Cohen, A. E. , Mahadevan, L. Kinks, Rings, and rackets in filamentous structures, *PNAS* 100(21) 12141-12146), shows carbon nanotubes which are spontaneously coiled and looped in tennis rocket form. This is a manifestation of potential instability of their straight configuration.

The effect of fiber diameter is dominant! ⇒ nanofibers

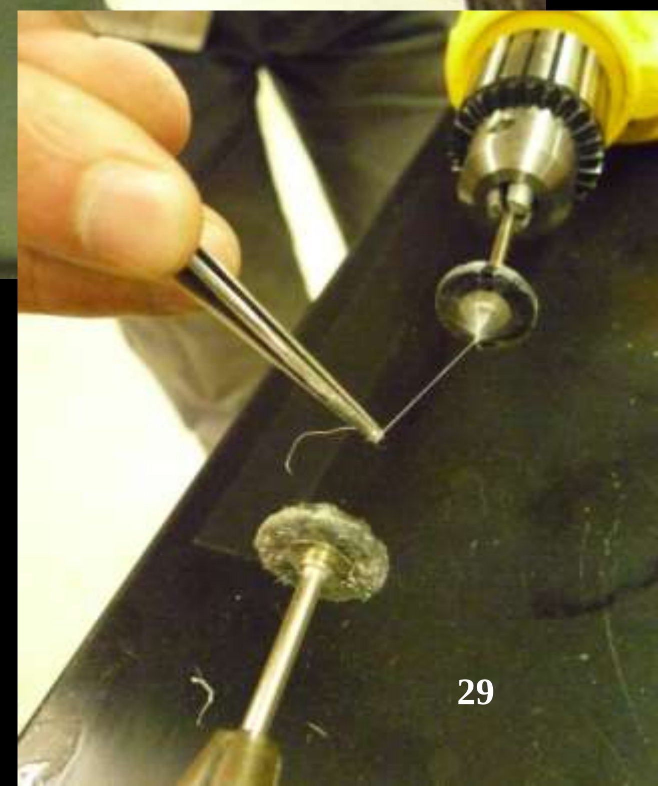
Carbon nanotubes







Nanoyarns



Flexibility of real fibers

Some examples of real fibers are collected in Table 7.1. The fiber characteristics are taken from

Chawla's book on "*High Performance Composites*"

and from

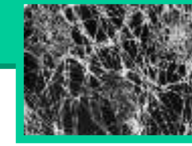
Feghelman's book on "*Mechanical Properties and Structure of Alpha-Keratin Fibers: Wool, Human Hair and Related Fibres*".

As you see, human hair gives sufficiently high flexibility!

Flexibility of real fibers

Table 7.1

	Diameter *d*	Young's modulus E	Flexibility f
E-glass	$14x10^{-6}$ m	70 E +9 Pa	7.5E+9 N^{-1} m^{-2}
PAN-based carbon, HM	$10x10^{-6}$ m	390 E +9 Pa	5.2E+9 N^{-1} m^{-2}
PAN-based carbon, HS	$8x10^{-6}$ m	250 E +9 Pa	1.9E+10 N^{-1} m^{-2}
Kevlar 49	$12x10^{-6}$ m	125 E +9 Pa	7.8E+9 N^{-1} m^{-2}
Boron	$100x10^{-6}$ m	385 E +9 Pa	5E+5 N^{-1} m^{-2}
Hair	$70x10^{-6}$ m	2 E +9 Pa	4.2E+8 N^{-1} m^{-2}

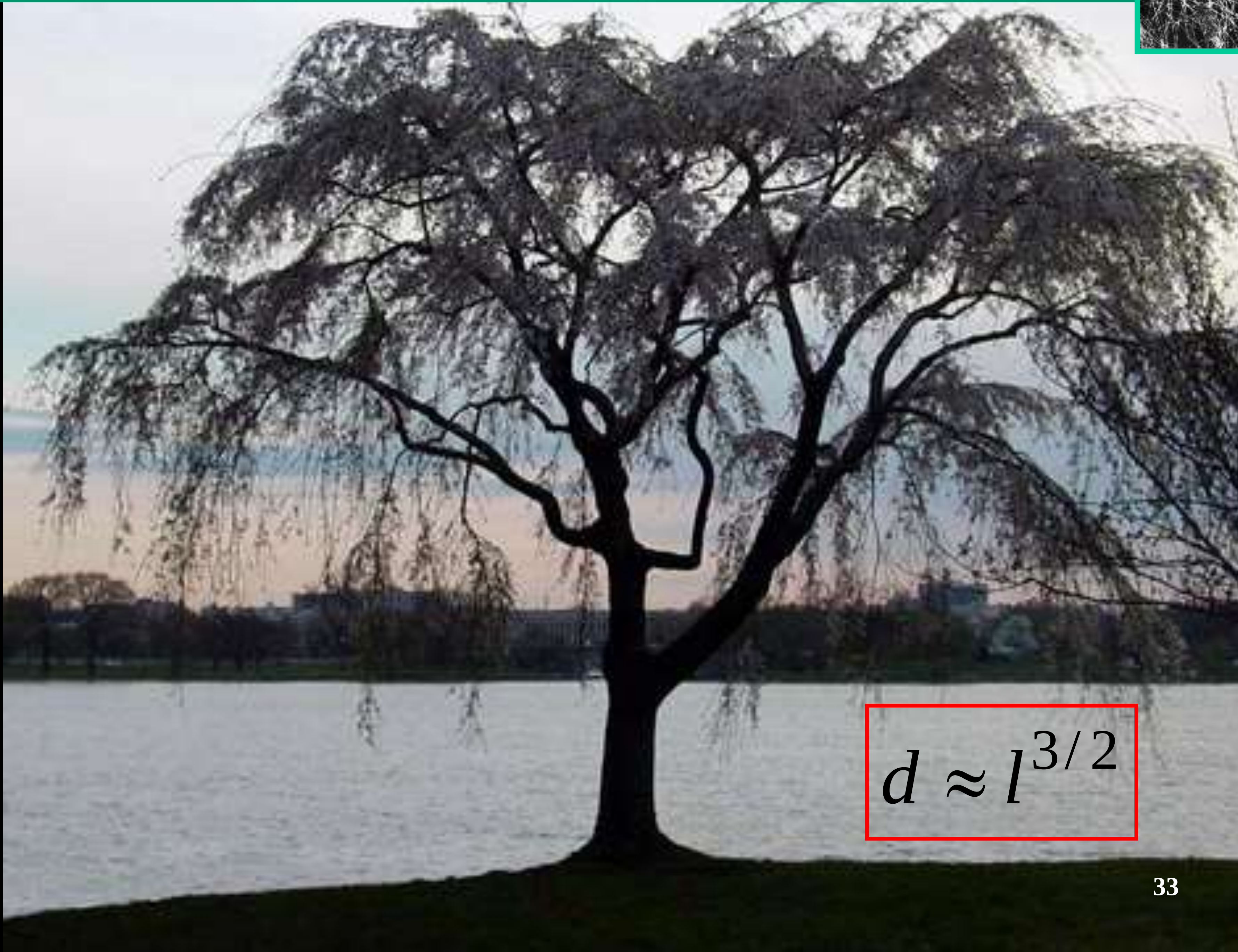


Rashevsky’s law

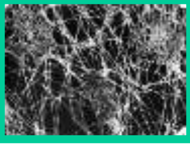
One interesting application of the Bernoulli-Euler beam theory has been suggested by Nikolas Rashevsky (*Mathematical biophysics; physicomathematical foundations of biology* Volume 2, p.251-255).

He was challenged by seeing that the longer the branch in a tree is growing, the farther it reaches out to the side, away from the trunk. But this trend continues only up to some critical length. There is a limit in branch length which controls how far it reaches out of the trunk and thus out of the shade of the highest branches.

Since the branch is elastic, and in a first approximation it can be modeled as a beam, Rashevsky provided some scaling estimates of this critical length when the branch can support itself without falling like branches in willows.



$$d \approx l^{3/2}$$



Rashevsky's law

Consider a branch of length *l* having a diameter *d* and apply Bernoulli-Euler formula

$$M = EI/R$$

to this branch. The bending moment on the branch tip is

$$M = Fl,$$

and if we assume that the weight is concentrated at the tip, we can put

$$F = \pi\rho g d^2 l/4,$$

where ρ is the wood density.

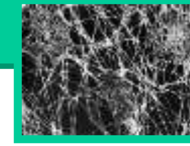
$$M = \frac{\pi\rho g d^2 l}{4} l = \frac{EI}{R}$$

The radius of curvature R which the branch can produce scales as

$$1/R \sim y/l^2$$

(Remember the formula for curvature $1/R \sim d^2Y/dX^2$.)

The sag (=průhyb) is denoted as y



Rashevsky's law

Hence, the sag (=průhyb) y gives an order of magnitude estimate for the beam deflection Y , and the branch length gives the order of magnitude estimate for X).

$$1/R \sim d^2Y/dX^2 \sim y/l^2 \qquad M = Fl = \frac{EI}{R} \Rightarrow \frac{1}{R} = \frac{Fl}{EI}$$

Thus,

$$y \sim Fl^{\,3}/EI \sim Fl^{\,3}/Ed^4. \qquad I = \frac{\pi d^4}{64}$$

This gives the sag per branch length T, that is konstant.

$$T = y/l \sim \rho g l^{\,3}/Ed^2\ ! \qquad F \sim \rho g d^2 l$$

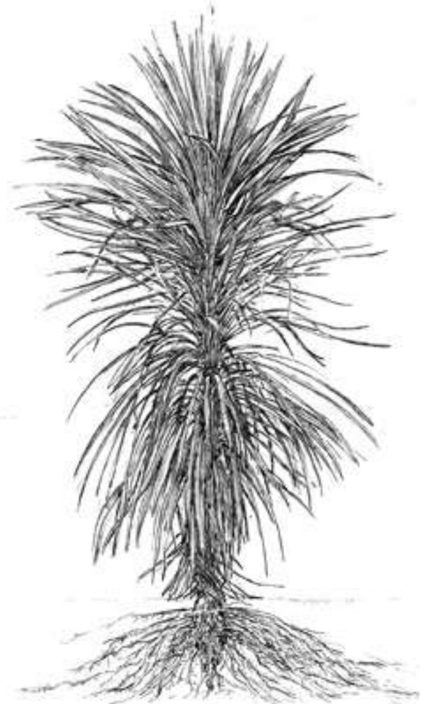
$$T \approx \frac{l^3}{d^2} \qquad d \approx l^{3/2}$$

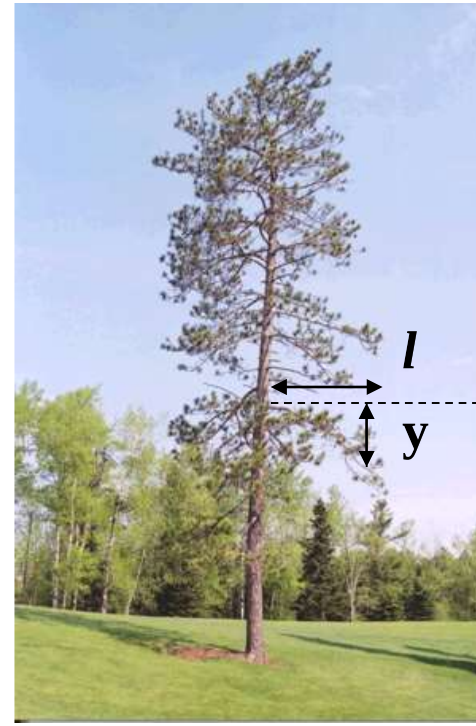
Rashevsky’s law

This elegant derivation was so attractive to biologists that they took a challenge and measured several whole trees including white oak with more than 3000 segments linking to various branch points!

McMahon & Bonner reported that local diameter was proportional to the 3/2 power of length to the tip as predicted by Rashevsky!

Rashevsky's law





$M = EI/R \Rightarrow M \sim Pl,\ R \sim y/l^2 \Rightarrow y \sim Pl^{3}/EI \sim Pl^{3}/Ed^4$

Since the branch bends under its own weight, $P \sim \rho g\, d^2\, l$

$\Rightarrow y \sim \rho g l^{4}/Ed^2 \Rightarrow$ **sag per branch length** $T \sim \rho g l^{3}/Ed^2$

$\Rightarrow$ **E for wood is proportional to density,** $E \sim \rho \Rightarrow$ $\boxed{T \sim l^{3}/d^2}$

McMahon& Bonner, "On size and life", 1983

Homework:

1. Měřením na stromech, keřích a rostlinách ověřte Rashevskeho zákon.

2. Jakou flexibilitu by měl vlas o průměru 100 nm?