Integrální počet

Primitivní funkce

$f(x) = 2x$

$F(x)$ ... primitivní fce k $f(x)$

$F'(x) = f(x)$ na I

$F(x) = x^2$

$F(x) = x^2 + 2$

$F(x) = x^2 + 100$

Neurčitý integrál

... množina všech primitivních fcí

$\int f(x)\,dx = F(x) + C,$

$C \in \mathbb{R}$

Integrační techniky

a) přímá metoda (integrace rozkladem, tabulkové integrace)

b) per-partes (po částech)

c) substituce

Linearita integrálu (neurčitého)

$$\int (f \pm g)\,dx = \int f\,dx \pm \int g\,dx \quad \ldots \text{ aditivita}$$

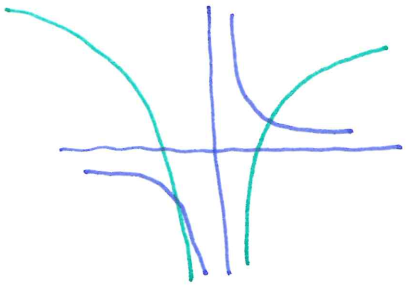
$$\int c \cdot f\,dx = c \cdot \int f\,dx \quad \ldots \text{ homogenita}$$

cvg

1a $\int \left(x^3 - \frac{1}{x} + \frac{\sqrt[4]{x}}{2}\right) dx = \int \left(x^3 - \frac{1}{x} + \frac{1}{2} \cdot x^{1/4}\right) dx$

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$

$\int \frac{1}{x} \, dx = \ln|x| + C$



$= \int x^3 \, dx - \int \frac{1}{x} \, dx + \frac{1}{2} \int x^{1/4} \, dx$

$= \frac{x^4}{4} - \ln|x| + \frac{1}{\cancel{2}} \cdot \frac{x^{\frac{5}{4}}}{\frac{5}{\cancel{4}\,2}} =$

$= \frac{x^4}{4} - \ln|x| + \frac{2}{5} \sqrt[4]{x^5} + C$

$C \in \mathbb{R}$

1c

$$\int \frac{x^3 - 2x + 1}{x^3}\,dx = \int \frac{x^3}{x^3}\,dx - \int \frac{2x}{x^3}\,dx + \int \frac{1}{x^3}\,dx$$

$$= \int 1\,dx - \int \frac{2}{x^2}\,dx + \int \frac{1}{x^3}$$

$$= \int 1\,dx - 2\int x^{-2}\,dx + \int x^{-3}\,dx$$

$$= x - 2\cdot\frac{x^{-1}}{-1} + \frac{x^{-2}}{-2}$$

$$= x + \frac{2}{x} - \frac{1}{2x^2} + C \quad , \; C \in \mathbb{R}$$

(1f)

$$\int \left(\sqrt{2x} + \sqrt{\frac{2}{x}} \right) dx$$

$$= \sqrt{2} \int \sqrt{x}\, dx + \sqrt{2} \int \frac{1}{\sqrt{x}}\, dx$$

$$= \sqrt{2} \int x^{1/2}\, dx + \sqrt{2} \int x^{-1/2}\, dx$$

$$= \sqrt{2} \cdot \frac{x^{3/2}}{3/2} + \sqrt{2} \cdot \frac{x^{1/2}}{1/2} = \sqrt{2} \left(\frac{2}{3} \sqrt{x^3} + 2 \sqrt{x} \right) + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x}\, dx = \operatorname{tg} x + C$$

$$\int \frac{1}{\sin^2 x}\, dx = -\operatorname{cotg} x + C$$

(1h)

$$\int \left(\sin x - \frac{1}{\cos^2 x} \right) dx = -\cos x - \operatorname{tg} x + C$$

1.

$$\int \frac{3-2\,cotg^2 x}{\cos^2 x}\,dx = \int \frac{3}{\cos^2 x}\,dx - \int \frac{2\,cotg^2 x}{\cos^2 x}\,dx$$

$$3\int \frac{1}{\cos^2 x}\,dx - 2\int \frac{\frac{\cancel{\cos^2 x}}{\sin^2 x}}{\frac{\cancel{\cos^2 x}}{1}}\,dx$$

$$= 3\,tg\,x - 2\int \frac{1}{\sin^2 x}\,dx$$

$$\boxed{cotg^2 x = \frac{\cos^2 x}{\sin^2 x}}$$

$$= 3\,tg\,x - 2(-\,cotg\,x)$$

$$\underline{\underline{= 3\,tg\,x + 2\,cotg\,x}}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + C$$

1.j $$\int \frac{4}{\sqrt{4-4x^2}}\,dx = \int \frac{\cancel{4}^{\,2}}{\cancel{\sqrt{4}}\cdot\sqrt{1-x^2}}\,dx$$

$$= 2\int \frac{1}{\sqrt{1-x^2}}\,dx = 2\arcsin x + C$$

1.k $$\int \frac{2}{3+3x^2}\,dx = \frac{2}{3}\int \frac{1}{1+x^2}\,dx = \frac{2}{3}\operatorname{arctg} x + C$$

CV 10

Per partes (integrace)

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx$$

$\boxed{\text{cv9/2a}}$

$$\int x \cdot \cos x \, dx = \left| \begin{array}{ll} u = \cos x & v' = x \\ \downarrow \text{derivace} & \downarrow \text{integrace} \\ u' = -\sin x & v = \frac{x^2}{2} \end{array} \right|$$

$$= \cos x \cdot \frac{x^2}{2} - \underbrace{\int (-\sin x) \cdot \frac{x^2}{2} \, dx}$$

… špatný postup

Správny postup

$$\int x \cdot \cos x \, dx = \left| \begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{array} \right|$$

$$= x \cdot \sin x - \int 1 \cdot \sin x \, dx = x \cdot \sin x - (-\cos x)$$

$$= \underline{\underline{x \cdot \sin x + \cos x + C}}$$

Pravidlo:

$$a) \int P(x) \cdot \left\{ \begin{array}{l} \sin(x) \\ \cos(x) \\ e^{x} \end{array} \right\} dx = \left| \begin{array}{ll} u = P(x) & v' = \left\{ \begin{array}{l} \sin x \\ \cos x \\ e^{x} \end{array} \right\} \end{array} \right|$$

polynom

b)

$$\int P(x) \cdot \begin{Bmatrix} \ln x \\ \operatorname{arctg} x \\ \operatorname{arcctg} x \\ \arcsin x \\ \arccos x \end{Bmatrix} dx = \left| \begin{matrix} u = \ln x, \ \operatorname{arc} & v' = P(x) \\ u' = & \end{matrix} \right|$$

2b)

$$\int x^3 \cdot e^x \, dx = \left| \begin{matrix} u = x^3 & v' = e^x \\ u' = 3x^2 & v = e^x \end{matrix} \right| = x^3 \cdot e^x - \cancel{\int 3x^2 \cdot e^x \, dx}$$

$$-\int 3x^2 \cdot e^x \, dx = \left| \begin{matrix} u = 3x^2 & v' = e^x \\ u' = 6x & v = e^x \end{matrix} \right| = x^3 \cdot e^x -$$

$$-\left(3x^2 \cdot e^x - \int 6x \cdot e^x \, dx\right) = x^3 e^x - 3x^2 e^x + \int 6x \cdot e^x \, dx$$

$$\left| \begin{matrix} u = 6x & v' = e^x \\ u' = 6 & v = e^x \end{matrix} \right| = x^3 e^x - 3x^2 e^x + 6x \cdot e^x - \underbrace{\boxed{\int 6 \cdot e^x \, dx}}_{6e^x}$$

$$= e^x \left(x^3 - 3x^2 + 6x - 6 \right) + C$$

$$\int P(x) \cdot \left\{ \begin{matrix} e^x \\ \cos x \\ \sin x \end{matrix} \right\} dx$$ ... metoda per-partes se použije opakovaně tolikrát, kolik je stupeň $P(x)$

1d $\int \ln x \, dx = \int 1 \cdot \ln x \, dx = \left| \begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{array} \right|$

$= x \cdot \ln x - \int \underbrace{\frac{1}{x} \cdot x}_{\int 1 dx = x} dx = \underline{\underline{x \cdot \ln x - x + C}}$

(RF) $\overbrace{\boxed{\int \cos^2 x \, dx}}^{= I} = \int \cos x \cdot \cos x \, dx = \left| \begin{array}{ll} u = \cos x & v' = \cos x \\ u' = -\sin x & v = \sin x \end{array} \right|$

$= \cos x \cdot \sin x - \int (-\sin x) \cdot \sin x \, dx$

$= \cos x \cdot \sin x + \int \sin^2 x \, dx = \cos x \cdot \sin x + \int 1 - \cos^2 x \, dx$

$= \cos x \cdot \sin x + \underbrace{\int 1 \, dx}_{= x} - \underbrace{\boxed{\int \cos^2 x \, dx}}_{I}$ ...integrální rovnice

$\cos^2 x + \sin^2 x = 1$

$$I = \cos x \cdot \sin x + x - I$$

$$2I = \cos x \cdot \sin x + x \quad | : 2$$

$$\underline{\underline{I = \frac{\cos x \cdot \sin x + x}{2} + C}}$$