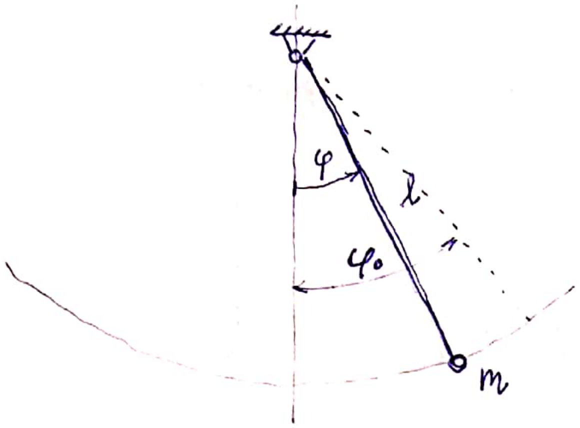
MATHEMATICAL PENDULUM

1



Given:

m ... mass of particle

l ... lenth of string

initial conditions

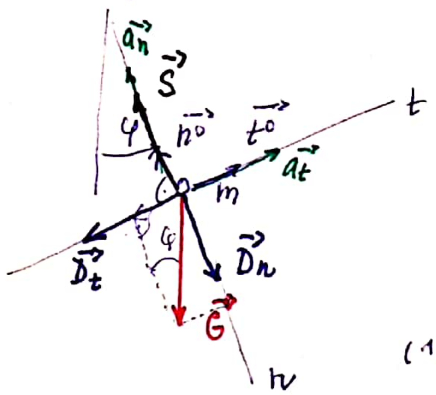
$\varphi(t=0)=\varphi_0$

$\dot{\varphi}(t=0)=0 \qquad \dot{\varphi}=\frac{d\varphi}{dt}$

Task:

$\varphi(t)$

Free body diagram



we use TNB coordinate system

Generally: $\sum_{(i)} \vec{F}_i + \vec{D} = \vec{0}$

$\vec{S} + \vec{G} + \vec{D} = \vec{0}$ vector equ. of dynamic equilibrium

component equs:

(1) t: $D_t + G \cdot \sin\varphi = 0$

(2) n: $D_n + G\cos\varphi - S = 0$

force specification:

(3) $G = mg$

(4) $D_t = m \cdot a_t$

(5) $D_n = m \cdot a_n$

(6) $a_t = l \cdot \ddot{\varphi}$

(7) $\underbrace{a_n = l \cdot \dot{\varphi}^2}_{\text{rotational motion}}$

(8) $\ddot{\varphi} = \frac{d^2\varphi}{dt^2}$ ... angular acceleration

(9) $\dot{\varphi} = \frac{d\varphi}{dt}$ ... angular velocity

substitution in to (1) and (2):

$$\boxed{(1) \quad \cancel{m} \cdot l \cdot \ddot{\varphi} + \cancel{m} g \cdot \sin\varphi = 0}$$

$$(2) \quad m \cdot l\,\dot{\varphi}^2 + mg\cos\varphi - S = 0$$

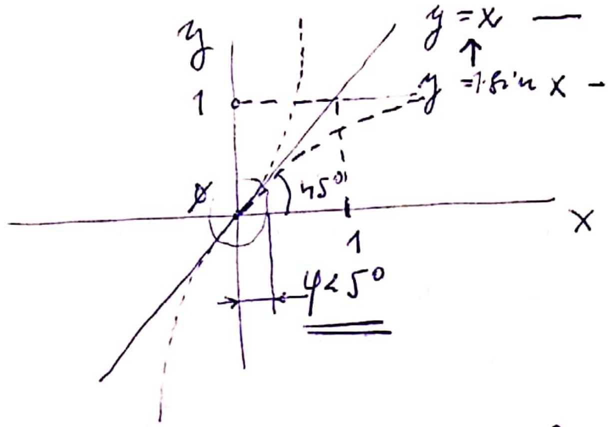
$$\boxed{\ddot{\varphi} + \frac{g}{l} \cdot \sin\varphi = \emptyset}$$ principal equation of motion (1DoF)

$\Downarrow$ $\varphi(t)$

differential eqn. of 2^{nd} order, homogeneous

<u>nonlinear</u> ($\sin\varphi$)

<u>linearisation:</u> $\sin\varphi \sim \varphi$ [rad]



($tg\,\varphi \sim \varphi$ [rad])

consideration: $\varphi < 5°$ $\Rightarrow \sin\varphi \sim \varphi$ [rad]

$$\boxed{\ddot{\varphi} + \frac{g}{l}\varphi = \emptyset}$$ linear diff. eqn.

<u>Solution:</u>

characteristic function

$$\lambda^2 + \frac{g}{l} = \emptyset$$

$$\lambda^2 = -\frac{g}{l}$$

$$\lambda_{1,2} = \pm\sqrt{-\frac{g}{l}} = \pm i\sqrt{\frac{g}{l}}$$

$$\lambda_1 = i\sqrt{\frac{g}{l}} \; ; \; \lambda_2 = -i\sqrt{\frac{g}{l}}$$

fundamental system

$e^{\lambda_1 t}$; $e^{\lambda_2 t}$

solution:

$$\varphi(t) = C_1 \cdot e^{i\sqrt{\frac{g}{l}}t} + C_2\, e^{-i\sqrt{\frac{g}{l}} \cdot t}$$

initial conditions $\Rightarrow C_1, C_2$

$$\dot{\varphi} = \frac{d\varphi}{dt} = C_1 i\sqrt{\frac{g}{l}}\, e^{i\sqrt{\frac{g}{l}}t} - C_2 i\sqrt{\frac{g}{l}}\, e^{-i\sqrt{\frac{g}{l}}t}$$

(*) $\varphi(t=\emptyset) = \varphi_0$

(**) $\dot{\varphi}(t=\emptyset) = \emptyset$

(*) $\varphi_0 = C_1 + C_2$

(**) $\emptyset = C_1 \cancel{i\sqrt{\frac{g}{l}}} - C_2 \cancel{i\sqrt{\frac{g}{l}}}$

(*) $\varphi_0 = C_1 + C_2 \Rightarrow$

(**) $\emptyset = C_1 - C_2 \Rightarrow C_1 = C_2 = \frac{\varphi_0}{2}$

Solution:

$$\boxed{\varphi(t) = \frac{\varphi_0}{2}\, e^{i\sqrt{\frac{g}{l}}t} + \frac{\varphi_0}{2}\, e^{i\sqrt{\frac{g}{l}}t}}$$

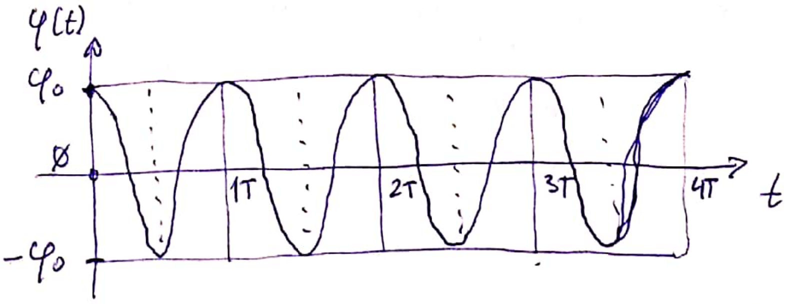
simplification: $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$

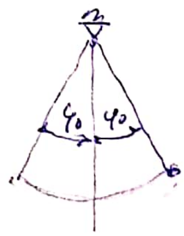
$e^{i\sqrt{\frac{g}{l}}t} = \cos\left(\sqrt{\frac{g}{l}}t\right) + i\sin\left(\sqrt{\frac{g}{l}}t\right)$ even function: $\cos(-\alpha) = \cos(\alpha)$

$e^{-i\sqrt{\frac{g}{l}}t} = \cos\left(-\sqrt{\frac{g}{l}}t\right) + i\sin\left(-\sqrt{\frac{g}{l}}t\right)$ odd function: $\sin(-\alpha) = -\sin(\alpha)$

$$\varphi(t) = \frac{\varphi_0}{2}\left\{\cos\left(\sqrt{\frac{g}{l}}t\right) + i\sin\left(\sqrt{\frac{g}{l}}t\right)\right\} + \frac{\varphi_0}{2}\left\{\cos\sqrt{\frac{g}{l}}t - i\sin\left(\sqrt{\frac{g}{l}}t\right)\right\} =$$

$$= \frac{\varphi_0}{2}\left\{2\cos\sqrt{\frac{g}{l}}t\right\} = \varphi_0 \cos\sqrt{\frac{g}{l}}t$$





$T = ?$

$$\boxed{\ddot{\varphi} + \frac{g}{l}\varphi = \emptyset}$$

Ω^2 ..... natural frequency (angular)

$\Omega = \sqrt{\frac{g}{l}}$ [rad/s] $\quad (\Omega = 2\pi f)$

frequency [Hz]

$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ [Hz]

period $\qquad f = \frac{1}{T}$

$$T = \frac{1}{f} = \frac{2\pi}{\sqrt{\frac{g}{l}}} \text{ [s]}$$