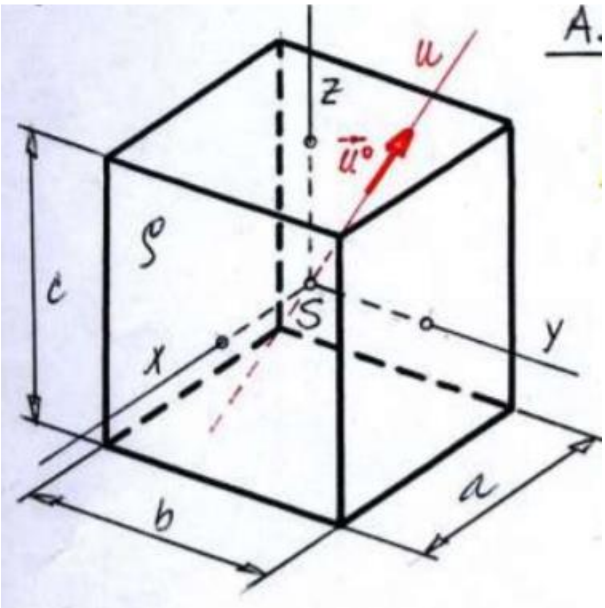
Moment of inertia of simple bodies

1. Rectangular prism

Given: a homogeneous rectangular prism with the mass density ρ and the dimensions a,b,c (as shown in the Figure)



Task:

Find the matrix of inertia of the prism to the coordinate system at the center of the prism (S,x,y,z).

Find the matrix of inertia of the prism with respect to the axis u which go through the center and a pole (as in the Figure)

Solution:

The matrix of inertia of the prism to the coordinate system

The volume of the elemental mass:

$$dm = \rho dV = \rho dxdydz \tag{1}$$

We calculate the moment of inertia with respect to the planes of the coordinate system:

$$\begin{aligned} J_{yz} &= \int_m x^2 dm = \rho \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} x^2 dxdydz \\ &= \rho \int_{-a/2}^{a/2} x^2 dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz = \rho \frac{1}{3}\left(\frac{a^3}{4}\right) bc = \frac{1}{12} a^2 m \end{aligned} \tag{2}$$

Because of the symmetry at the center point, so we get :

$$J_{zx} = \int_m y^2 dm = \frac{1}{12} b^2 m \quad (3)$$

$$J_{xy} = \int_m z^2 dm = \frac{1}{12} c^2 m \quad (4)$$

So we calculate the moment of inertia with respect to the axis :

$$J_x = J_{xy} + J_{xz} = \frac{1}{12}\left(b^2 + c^2\right) m \quad (5)$$

$$J_y = J_{yz} + J_{yx} = \frac{1}{12}\left(c^2 + a^2\right) m \quad (6)$$

$$J_z = J_{zx} + J_{zy} = \frac{1}{12}\left(a^2 + b^2\right) m \quad (7)$$

Next, we calculate the cross product moments of inertia:

$$D_{xy} = \int_m xydm = \rho \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} xydxdydz = \rho \int_{-a/2}^{a/2} xdx \int_{-b/2}^{b/2} ydy \int_{-c/2}^{c/2} dz = 0 \quad (8)$$

In the same way, we have:

$$D_{yz} = D_{zx} = 0 \quad (9)$$

So we have the matrix of inertia to the coordinate system:

$$I = \begin{bmatrix} \frac{1}{12}\left(b^2 + c^2\right) m & 0 & 0 \\ 0 & \frac{1}{12}\left(c^2 + a^2\right) m & 0 \\ 0 & 0 & \frac{1}{12}\left(a^2 + b^2\right) m \end{bmatrix} \quad (10)$$

Or

$$I = \frac{1}{12} m \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix} \quad (11)$$

The moments of inertia with respect to the axis *u*

Firstly, we find the unit vector of the axis u:

$$\vec{u}^0 = \frac{a}{u} \vec{i} + \frac{b}{u} \vec{j} + \frac{c}{u} \vec{k} \quad (12)$$

Where:

$$u = \sqrt{a^2 + b^2 + c^2}$$

We represent the unit vector in matrix form:

$$u^0 = \frac{1}{u}\begin{bmatrix} a \\ b \\ c \end{bmatrix}, u^{0T} = \frac{1}{u}\begin{bmatrix} a & b & c \end{bmatrix} \tag{13}$$

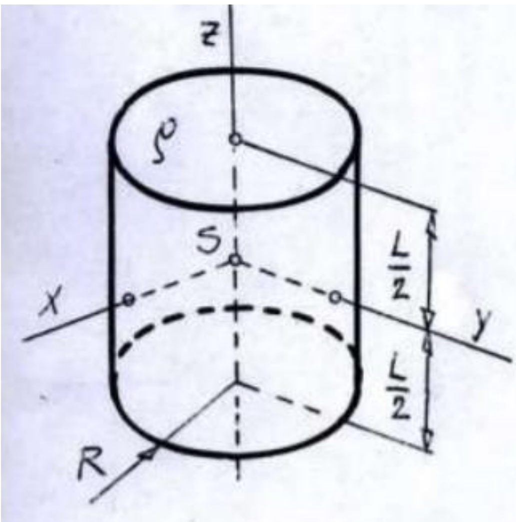
Then we have the moment of inertia with respect to the axis u:

$$J_u = u^{0T}.I.u^0 = \frac{m}{12u^2}\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{14}$$

$$= \frac{m}{12}.\frac{a^2\left(b^2 + c^2\right) + b^2\left(c^2 + a^2\right) + c^2\left(a^2 + b^2\right)}{a^2 + b^2 + c^2}$$

2. Cylinder

Given: a homogeneous cylinder with the mass density ρ and the dimensions: R (the radius), L (the length) (as shown in the Figure)

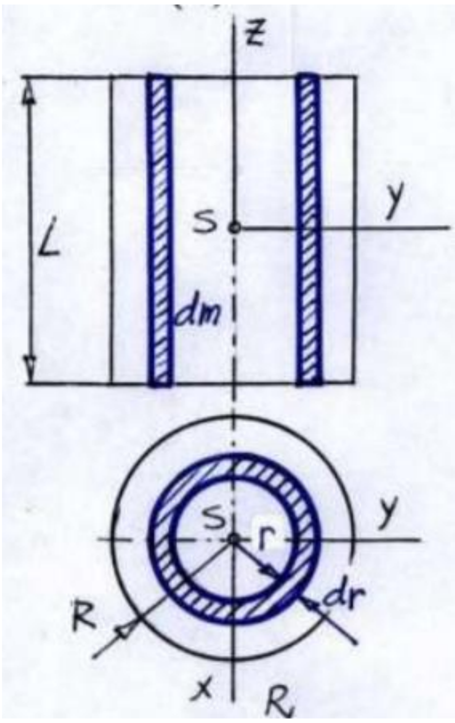


Task:

Find the matrix of inertia of the prism to the coordinate system at the center of the cylinder (S,x,y,z).

Solution:

For convenience in calculating the moment of inertia with respect to the axis z, we can analyze the element of mass following the scheme below:



According to the scheme , the volume of the elemental mass:

$$dm = \rho dV = \rho 2\pi Lrdr \quad (1)$$

And total mass:

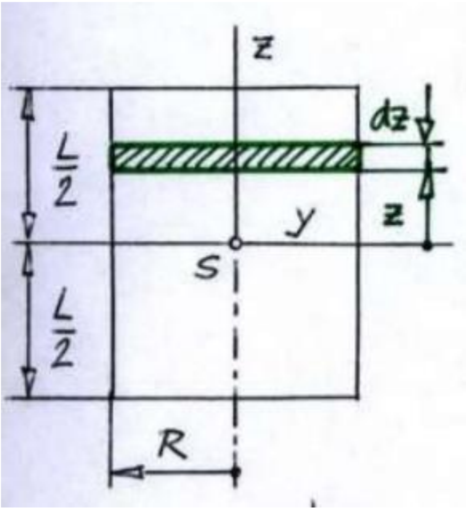
$$m = \rho V = \rho 2\pi LR^2 \quad (2)$$

We calculate the moment of inertia with respect to the axis z:

$$J_z = \int_m \left(x^2 + y^2\right) dm = \int_m \left[\left(r\cos\varphi\right)^2 + \left(r\sin\varphi\right)^2\right] dm = \int_m r^2 dm \quad (3)$$

$$= 2\pi\rho L \int_0^R r^3 dr = 2\pi\rho L \left(\frac{r^4}{4} \Bigg|_0^R \right) = \rho \frac{\pi R^4 L}{2} = \frac{1}{2} mR^2$$

Next, for calculating the moment of inertia with respect to the plane (Sxy), we analyze the element of mass as shown in Figure below:



According to the scheme , the volume of the elemental mass:

$$dm = \rho dV = \rho 2\pi R^2 dz \quad (4)$$

We calculate the moment of inertia with respect to the plane (Sxy):

$$J_{xy} = \int_m z^2 dm = 2\pi R^2 \int_{-L/2}^{L/2} z^3 dz = 2\pi R^2 \left(\frac{z^4}{4} \Bigg|_{-L/2}^{L/2} \right) = \rho \frac{\pi R^2 L^3}{12} = \frac{1}{12} mL^2 \quad (5)$$

Because of the symmetry, we get :

$$J_{zx} = J_{yz} = \frac{1}{2} J_z = \frac{1}{4} mR^2 \quad (6)$$

So we calculate the moment of inertia with respect to the axis :

$$J_x = J_y = J_{xy} + J_{xz} = \frac{1}{4} m \left(R^2 + \frac{L^2}{3} \right) \quad (7)$$

Next, also because of the symmetry, it is easy to see that the cross product moments of inertia:

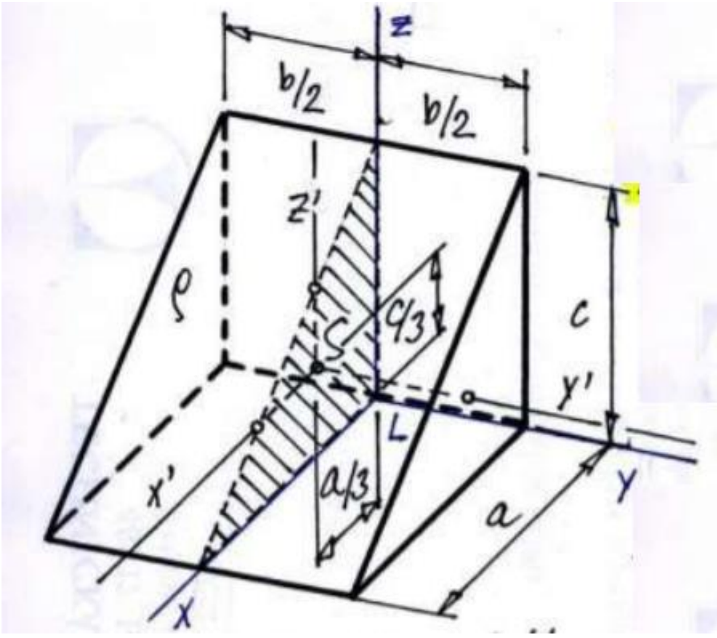
$$D_{xy} = D_{yz} = D_{zx} = 0 \tag{8}$$

So we have the matrix of inertia to the coordinate system:

$$I = \frac{1}{4}m\begin{bmatrix} R^2 + \frac{L^2}{3} & 0 & 0 \\ 0 & R^2 + \frac{L^2}{3} & 0 \\ 0 & 0 & 2R^2 \end{bmatrix} \tag{9}$$

3. Triangular prism

Given: a homogeneous prism with the mass density ρ and the dimensions of sides: a, b, c (as shown in the Figure)



Task:

Find the matrix of inertia of the prism to the coordinate system (*S,x',y',z'*).

Solution:

Firstly, we calculate the matrix of inertia of the prism to the original coordinate system (*L,x,y,z*)

We use the differential element (as shown in the Figure) for integration:

$$\int_{(V)} F(x,y,z)dV = \int_0^c \int_{-b/2}^{b/2} \int_c^{a\left(1-\frac{z}{c}\right)} F(x,y,z)dxdydz \tag{1}$$

The integration function F(x,y,z) have the limits of integration as following:

$$0 \le z \le c$$
$$-\frac{b}{2} \le y \le \frac{b}{2} \tag{2}$$
$$0 \le x \le a\left(1-\frac{z}{c}\right)$$

Using the integration (1), we calculate the moment of inertia with respect to the axises:

$$J_x = \rho\int_0^c\int_{-b/2}^{b/2}\int_c^{a\left(1-\frac{z}{c}\right)}\left(y^2+z^2\right)dxdydz = \rho\int_0^c\int_{-b/2}^{b/2}\left(y^2+z^2\right)\int_c^{a\left(1-\frac{z}{c}\right)}dxdydz \tag{3}$$

$$= \rho a\int_0^c\int_{-b/2}^{b/2}\left(y^2+z^2\right)\left(1-\frac{z}{c}\right)dydz = \rho a\int_0^c\int_{-b/2}^{b/2}\left(y^2+z^2-y^2\frac{z}{c}-\frac{z^3}{c}\right)dydz$$

$$= \rho a\int_0^c\left(\frac{y^3}{3}+z^2y-\frac{y^3}{3}\frac{z}{c}-\frac{z^3}{c}y\right)_{-b/2}^{b/2}dz = \rho a\int_0^c\left(\frac{b^3}{12}+bz^2-\frac{b^3}{12}\frac{z}{c}-b\frac{z^3}{c}\right)dz$$

$$= \rho a\left(\frac{b^3c}{12}+\frac{bc^3}{3}-\frac{b^3c}{24}-\frac{bc^3}{4}\right) = \rho\frac{abc}{24}\left(b^2+2c^2\right)$$

$$J_y = \rho\int_0^c\int_{-b/2}^{b/2}\int_c^{a\left(1-\frac{z}{c}\right)}\left(x^2+z^2\right)dxdydz = \rho a\int_0^c\int_{-b/2}^{b/2}\left[z^2\left(1-\frac{z}{c}\right)+\frac{a^2}{3}\left(1-\frac{z}{c}\right)^3\right]dydz \tag{4}$$

$$= \rho a\int_0^c\left(\frac{a^3}{3}-\frac{a^2z}{c}+\left(1-\frac{a^2}{c^2}\right)^3z^2-\left(\frac{1}{c}+\frac{a^2}{3c^2}\right)z^3\right)_{-b/2}^{b/2}dz$$

$$= \rho a\left[\frac{a^3c}{3}-\frac{ca^2}{2}+\left(1+\frac{a^2}{c^2}\right)\frac{c^2}{3}-\frac{1}{c}\left(1+\frac{a^2}{3c^2}\right)\frac{c^4}{4}\right]$$

$$= \rho\frac{abc}{12}\left(a^2+c^2\right)$$

$$J_z = \rho\int_0^c\int_{-b/2}^{b/2}\int_c^{a\left(1-\frac{z}{c}\right)}\left(x^2+y^2\right)dxdydz = \rho a\int_0^c\int_{-b/2}^{b/2}\left[\frac{a^2}{3}\left(1-\frac{z}{c}\right)^3+y^2\left(1-\frac{z}{c}\right)\right]dydz \tag{5}$$

$$= \rho a\int_0^c\left[\frac{a^2}{3}\left(1-\frac{a^2}{c^2}\right)^3y+\frac{y^3}{3}\left(1-\frac{z^2}{c^2}\right)\right]_{-b/2}^{b/2}dz$$

$$= \rho ab\int_0^c\left[\frac{a^2}{3}+\frac{b^2}{12}-\left(a^2+\frac{b^2}{12}\right)^3\frac{z}{c}+a^2\frac{z^2}{c^2}-\frac{a^2}{3}\frac{z^3}{c^3}\right]dz$$

$$= \rho abc\left[\left(\frac{a^2}{3}+\frac{b^2}{12}\right)-\frac{1}{2}\left(a^2+\frac{b^2}{12}\right)+\frac{a^2}{3}-\frac{a^2}{12}\right]$$

$$= \rho\frac{abc}{24}\left(b^2+2c^2\right)$$

Next, we calculate the cross moment of inertia with respect to the plane (*Lzx*).

$$D_{zx} = \rho \int_0^c \int_{-b/2}^{b/2} \int_c^{a\left(1-\frac{z}{c}\right)} zx dx dy dz = \rho \frac{a^2}{2} \int_0^c z\left(1-\frac{z}{c}\right)^2 \int_{-b/2}^{b/2} dy dz \tag{6}$$

$$= \rho \frac{a^2}{2} b \int_0^c \left[z - 2\frac{z^2}{c} + \frac{z^3}{c^2} \right] dz = \rho \frac{a^2}{2} bc^2 \left[\frac{1}{2} - 2\frac{1}{3} + \frac{1}{4} \right]$$

$$= \rho \frac{a^2 bc^2}{24}$$

Because of the symmetry, we get :

$$D_{xy} = D_{yz} = 0 \tag{7}$$

And it is easy to calculate the total mass:

$$m = \rho V = \rho \frac{abc}{2} \tag{8}$$

So we get:

$$J_x = \frac{1}{12} m\left(b^2 + 2c^2\right) \tag{9}$$

$$J_y = \frac{1}{6} m\left(c^2 + a^2\right)$$

$$J_z = \frac{1}{12} m\left(2a^2 + b^2\right)$$

$$D_{zx} = \frac{1}{12} mac$$

So we have the matrix of inertia to the coordinate system:

$$I = \frac{1}{12} m \begin{bmatrix} b^2 + 2c^2 & 0 & -ac \\ 0 & 2c^2 + 2a^2 & 0 \\ -ac & 0 & 2a^2 + b^2 \end{bmatrix} \tag{10}$$

Now, we calculate the matrix of inertia in the shifted coordinate system (*Sx'y'z'*).

Because S is the center of mass of the prism so in accordance with the Steiner's theorem, we have :

$$I = I' + D$$

Where:

I' is the matrix of inertia in the shifted coordinate system (*Sx'y'z'*).

D is the matrix of Steiner's components

For calculating the matrix D, firstl, we find the shifted vector from the coordinate system (*Sx'y'z'*) to coordinate system (*Lxyz*) is:

$$\vec{r} = -\frac{a}{3}\vec{i} + 0\vec{j} - \frac{c}{3}\vec{k} \tag{11}$$

We represent the vector in matrix form:

$$r = \begin{bmatrix} -\frac{a}{3} \\ 0 \\ -\frac{c}{3} \end{bmatrix} \tag{12}$$

Then we have the matrix of Steiner's components:

$$D = \frac{m}{9}\begin{bmatrix} c^2 & 0 & -ac \\ 0 & c^2 + a^2 & 0 \\ -ac & 0 & a^2 \end{bmatrix} \tag{13}$$

So the matrix of inertia in the shifted coordinate system (*Sx'y'z'*) is

$$I' = I - D = \frac{1}{12}m\begin{bmatrix} b^2 + 2c^2 & 0 & -ac \\ 0 & 2c^2 + 2a^2 & 0 \\ -ac & 0 & 2a^2 + b^2 \end{bmatrix} - \frac{m}{9}\begin{bmatrix} c^2 & 0 & -ac \\ 0 & c^2 + a^2 & 0 \\ -ac & 0 & a^2 \end{bmatrix} \tag{14}$$

$$= \frac{1}{36}m\begin{bmatrix} 3b^2 + 2c^2 & 0 & ac \\ 0 & 2c^2 + 2a^2 & 0 \\ ac & 0 & 2a^2 + 3b^2 \end{bmatrix}$$