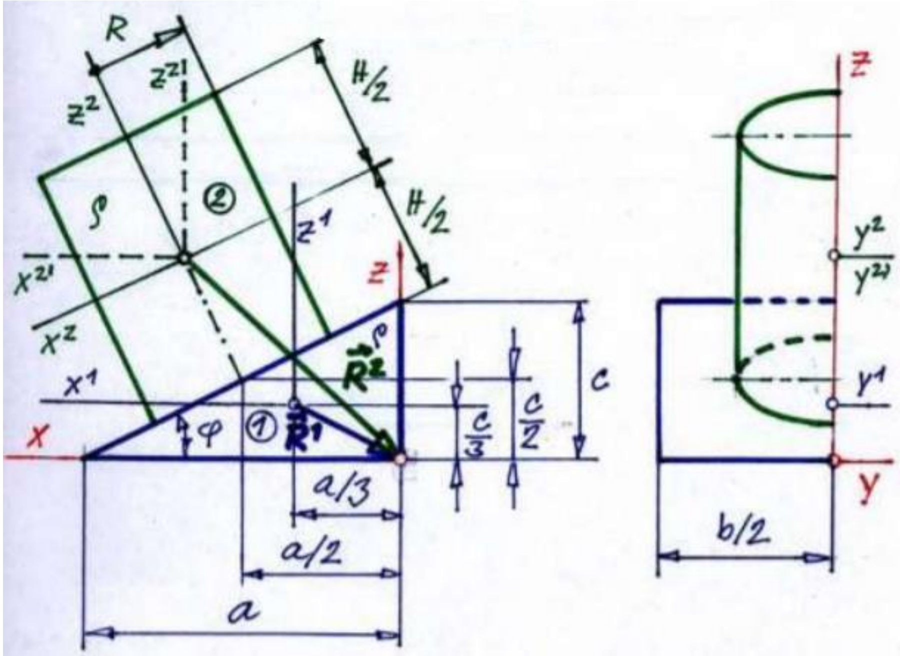
Moments of inertia of a system of bodies

Given: a system of two bodies as shown in the Figure below. The first body is a homogeneous prism with the mass density ρ and the dimensions of sides: a, b, c. The second one is a homogeneous cylinder with the mass density ρ and the dimensions: R (the radius), H (the length).



Task:

Find the matrix of inertia of the system (I) to the coordinate system (x,y,z).

Solution:

We get the mass of the prism from the previous exercise:

$$m_1 = \rho V_1 = \rho \frac{abc}{2} \tag{1}$$

And the matrix of inertia in the shifted coordinate system (x^1 y^1 z^1) is:

$$I^1 = \frac{1}{36} m_1 \begin{bmatrix} 3b^2 + 2c^2 & 0 & ac \\ 0 & 2c^2 + 2a^2 & 0 \\ ac & 0 & 2a^2 + 3b^2 \end{bmatrix} \tag{2}$$

The shifted vector from the coordinate system (x^1 y^1 z^1) to coordinate system (xyz) is:

$$\vec{r} = -\frac{a}{3}\vec{i} + 0\vec{j} - \frac{c}{3}\vec{k} \tag{1}$$

Then we have the matrix of Steiner's components:

$$D^1 = \frac{m_1}{9}\begin{bmatrix} c^2 & 0 & -ac \\ 0 & c^2+a^2 & 0 \\ -ac & 0 & a^2 \end{bmatrix} \quad (2)$$

So we have the matrix of inertia of the prism to the coordinate system (*xyz*):

$$I^1_{xyz} = I^1 + D^1 = \frac{1}{12}m_1\begin{bmatrix} b^2+2c^2 & 0 & -ac \\ 0 & 2c^2+2a^2 & 0 \\ -ac & 0 & 2a^2+b^2 \end{bmatrix} \quad (3)$$

We get the mass of the cylinder from the previous exercise:

$$m_2 = \rho V_2 = \rho\pi R^2 H \quad (4)$$

And the matrix of inertia in the coordinate system ($x^2\ y^2\ z^2$) is:

$$I^2 = \frac{1}{4}m_2\begin{bmatrix} R^2+\frac{H^2}{3} & 0 & 0 \\ 0 & R^2+\frac{H^2}{3} & 0 \\ 0 & 0 & 2R^2 \end{bmatrix} \quad (5)$$

From the Figure, we see that the coordinate system (*x y z*) can be obtained by rotating the coordinate system ($x^2\ y^2\ z^2$) with rotated angle φ about the axis y^2 then translating it with translated vector $\vec{r}_2$, where:

$$\varphi = acrtg\frac{c}{a} \quad (6)$$

And

$$\vec{r}_2 = -\left(\frac{H}{2}\sin\varphi + \frac{a}{2}\right)\vec{i}_2 + 0\vec{j}_2 - \left(\frac{H}{2}\cos\varphi + \frac{c}{2}\right)\vec{k}_2 \quad (7)$$

So we have the matrix of transformation for rotation about axis y^2 is:

$$T = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \quad (8)$$

Then we find the matrix of inertia of body 2 in the coordinate system ($x^{2'}\ y^{2'}\ z^{2'}$) is:

$$I^{2'} = T^T I^2 T \quad (9)$$

And we have the matrix of Steiner's components:

$$D^2 = \begin{bmatrix} D^2_{1,1} & D^2_{1,2} & D^2_{1,3} \\ D^2_{2,1} & D^2_{2,2} & D^2_{2,3} \\ D^2_{3,1} & D^2_{3,2} & D^2_{3,3} \end{bmatrix} \tag{10}$$

Where:

$$D^2_{1,1} = m_2\left(\frac{H^2}{4}\cos^2\varphi + \frac{Hc}{2}\cos\varphi + \frac{c^2}{4}\right) \tag{11}$$

$$D^2_{2,2} = m_2\left(\frac{H^2}{4} + \frac{H}{2}(c\cos\varphi + a\sin\varphi) + \frac{c^2 + a^2}{4}\right)$$

$$D^2_{3,3} = m_2\left(\frac{H^2}{4}\sin^2\varphi + \frac{Ha}{2}\sin\varphi + \frac{a^2}{4}\right)$$

$$D^2_{1,3} = D^2_{3,1} = \frac{m_2}{4}\left(H^2\sin\varphi\cos\varphi + Hc\sin\varphi + Ha\cos\varphi + ac\right)$$

$$D^2_{1,2} = D^2_{2,1} = D^2_{2,3} = D^2_{3,2} = 0$$

So we have the matrix of inertia of the cylinder to the coordinate system (*xyz*):

$$I^2_{xyz} = I^{2'} + D^2 \tag{12}$$

The matrix of inertia of the system (I) to the coordinate system (*xyz*) is:

$$I = I^1_{xyz} + I^2_{xyz} \tag{13}$$